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On Dynamics of Fractional Order Dual Lorenz System

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Abstract

This article investigates the dynamical properties of a novel fractional order dual Lorenz system (FDLS) for various parameters. The dynamical behavior of the system is examined through Lyapunov exponents, and the behavior near equilibrium points. Additionally, using the mathematical software MATLAB, we provide graphical visualizations of the dual Lorenz system for selected parameter values to illustrate its complex dynamics.

Keywords: Chaos, fractional derivative, stability, Lyapunov exponent, FDLS

1 Introduction

The Lorenz system of differential equations belongs to the class of chaotic systems. As a nonlinear autonomous dynamical system, the Lorenz system has been extensively studied in the mathematical literature. It is typically represented as a set of three coupled differential equations. The qualitative properties of systems are often analyzed through graphical visualizations (see [1–5] and the references cited in).

The dual Lorenz system is a mathematical model derived by modifying the Lorenz system to explore more complex dynamical behaviors, such as synchronization, chaos control, and bifurcations [6]. It is a subsystem of a system which is obtained from Lorenz system by replacing the real variable by complex variables. It often involves two interacting sets of Lorenz equations, where each system influences the other through specific coupling terms. This dual configuration enables researchers to study the mutual influence of chaotic systems, analyze stability under varying conditions, and investigate applications in secure communications, weather modeling, and neural networks. The dual Lorenz system is particularly useful in understanding the dynamics of interconnected chaotic systems and provides a valuable framework for testing control strategies and synchronization techniques. Dynamic analysis of the dual Lorenz system is discussed for integer order system

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by Zlatanovska and Piperevski [6, 7].

Motivated by the above discussion we have proposed a novel fractional order dual Lorenz system (FDLS) and studied some of its properties. Some more results for the fractional order system can be found in the papers [8–11] and in the references cited in these. In a recent study, Srivastava et al.[12] discussed the function projective synchronization of chaotic systems using a tracking control scheme.
In this paper an attempt is made to study the chaos in the novel fractional order dual Lorenz system. The Benettin-Wolf algorithm based MATLAB code has been used for the calculation of Lypunaov exponent [13].
The structure of this article is as follows. The basic definition of fractional calculus is provided in Section 2. In Section 3, a novel fractional order dual Lorenz system is proposed, and stability of equilibrium points is examined. The numerical simulations are provided in Section 4 to determine whether chaos exists. In Section 5, a final statement is provided.

2 Basic of Fractional Calculus

The definition of the Caputo fractional derivative is given in this section. The fractional derivative D_t^α of $f(t)$, in the Caputo sense is defined as [14–18]

$$D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\xi)^{n-\alpha-1} f^n(\xi) d\xi, \quad (n-1 < \alpha < n).$$

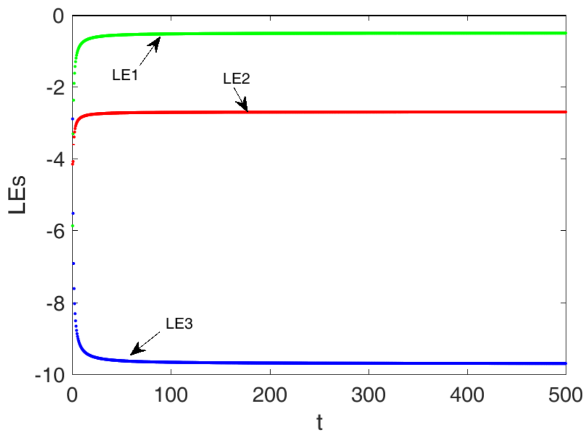
The main advantage of Caputo's definition, compared to the Riemann-Liouville definition, lies in its compatibility with conventional initial conditions, which are more straightforward to interpret and apply. Additionally, while the Caputo derivative of a constant is zero (i.e., bounded), the Riemann-Liouville derivative of a constant becomes unbounded at t=0 [14].

Table 1: Time evolution of the LEs for the FDLS for $\alpha = 0.99$.

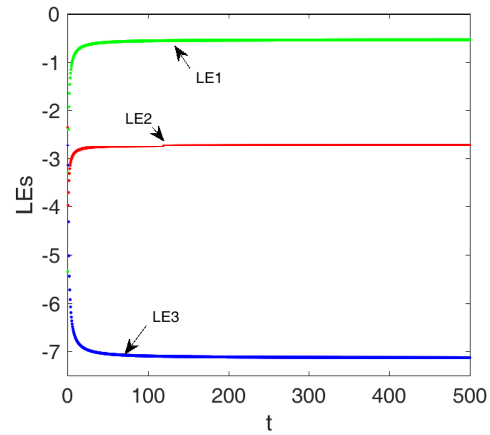
t	LE-1	LE-2	LE-3
50.00	-0.5358	-2.7012	-9.6057
100.00	-0.5078	-2.6872	-9.6475
150.00	-0.4984	-2.6826	-9.6614
200.00	-0.4938	-2.6803	-9.6684
250.00	-0.4910	-2.6789	-9.6726
300.00	-0.4891	-2.6780	-9.6754
350.00	-0.4878	-2.6773	-9.6773
400.00	-0.4868	-2.6768	-9.6788
450.00	-0.4860	-2.6764	-9.6800
500.00	-0.4854	-2.6761	-9.6809

3 System Description and stability analysis

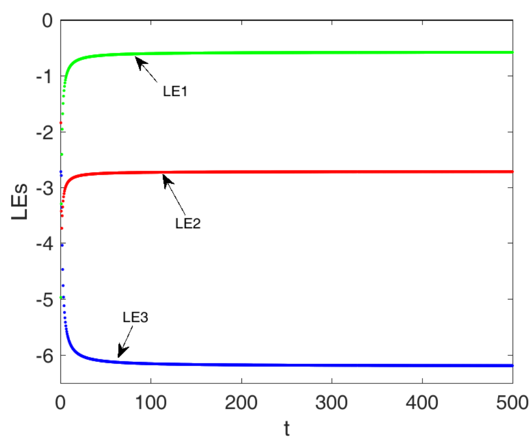
This section proposes a novel fractional order system. The system's equilibrium points are then determined, and their stability is discussed. The novel FDLS is given by



(a) Plot of the LEs vs time for the FDLS for $\alpha = 0.99$



(b) Plot of the LEs vs time for the FDLS for $\alpha = 0.9$



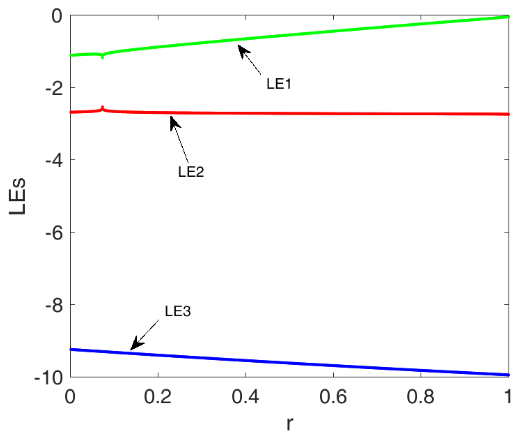
(c) Plot of the LEs vs time for the FDLS for $\alpha = 0.8$

Fig. 1: Comparison of the Lyapunov exponents verses t for different values of the fractional-order parameter α.

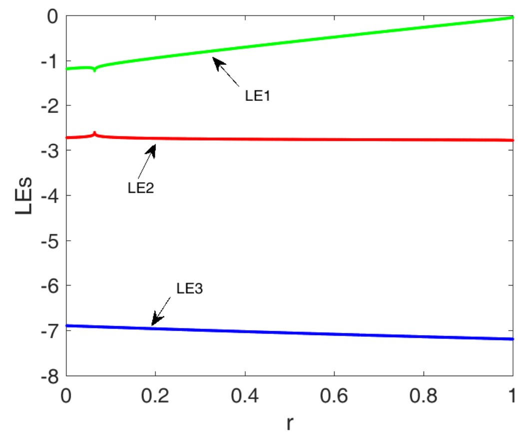
$$\begin{aligned}
\frac{d^{q_1}x}{dt^{\alpha}} &= \sigma(y-x) \\
\frac{d^{q_2}x}{dt^{\alpha}} &= x(r-z)-y \\
\frac{d^{q_3}x}{dt^{\alpha}} &= -xy-bz, \qquad \sigma, r, b > 0,
\end{aligned} \tag{1}$$

where $0 < q_1, q_2, q_3 \leq 1$, and fractional derivatives are considered in the Caputo sense. Without loss of generality, let us assume that

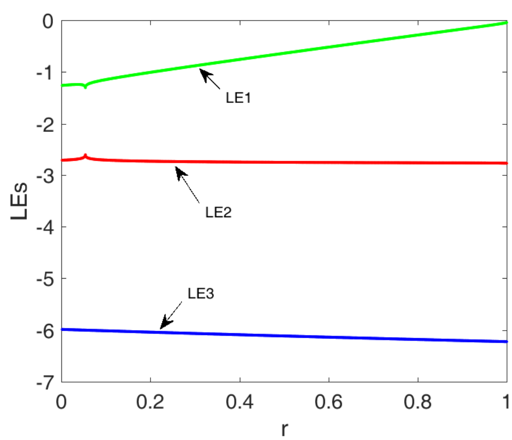
$$q_1 = q_2 = q_3 = \alpha.$$



(a) Plot of the LEs vs r for the FDLS for $\alpha = 0.99$



(b) Plot of the LEs vs r for the FDLS for $\alpha = 0.9$



(c) Plot of the LEs vs r for the FDLS for $\alpha = 0.8$

Fig. 2: Comparison of the Lyapunov exponents with respect to the system parameter r for different values of the fractional-order parameter α.

So, equation (1) can be written as

$$\begin{aligned} \frac{d^\alpha x}{dt^\alpha} &= \sigma(y - x) \\ \frac{d^\alpha x}{dt^\alpha} &= x(r - z) - y \\ \frac{d^\alpha x}{dt^\alpha} &= -xy - bz, \qquad \sigma, r, b > 0. \end{aligned} \tag{2}$$

To find the equilibrium points of system (2), we have to solve following equation:

$$\begin{aligned} \sigma(y-x) &= 0 \\ x(r-z)-y &= 0 \\ -xy-bz &= 0. \end{aligned}$$

Case-I (when $r > 1$)
In this case, we get only one equilibrium point $E_1(0,0,0)$.
Case-II(when $0 < r < 1$)
In this case, we get three equilibrium points $E_1(0,0,0)$, $E_2(\sqrt{b(1-r)}, \sqrt{b(1-r)}, r-1)$ and $E_3(-\sqrt{b(1-r)}, -\sqrt{b(1-r)}, r-1)$.
The Jacobian matrix for the system (2) at an equilibrium point $\bar{E}(\bar{x},\bar{y},\bar{z})$ is given as

$$J(\bar{E}) = \begin{pmatrix} -\sigma & \sigma & 0 \\ r-\bar{z} & -1 & -\bar{x} \\ -\bar{y} & -\bar{x} & -b \end{pmatrix} \tag{3}$$

Table 2: Time evolution of the LEs for the FDLS for $\alpha = 0.9$.

t	LE-1	LE-2	LE-3
50.00	-0.5775	-2.7391	-7.0402
100.00	-0.5498	-2.7243	-7.0826
150.00	-0.5405	-2.7194	-7.0967
200.00	-0.5359	-2.7169	-7.1037
250.00	-0.5331	-2.7154	-7.1080
300.00	-0.5312	-2.7144	-7.1108
350.00	-0.5299	-2.7137	-7.1128
400.00	-0.5289	-2.7132	-7.1143
450.00	-0.5281	-2.7128	-7.1155
500.00	-0.5275	-2.7124	-7.1165

Lemma 1. *For all $0 < \alpha < 1$, the equilibrium point $E_1(0,0,0)$ is stable when $0 < r < 1$ and unstable when $r > 1$.*

Proof. The characteristic equation of the Jacobian matrix (3) for equilibrium point E_1 is given by

$$(b+\lambda)(\lambda^2 + (1+\sigma)\lambda + (1-r)\sigma) = 0 \tag{4}$$

The eigenvalues of the Jacobian matrix (3) are given by $\lambda_1 = -b$, $\lambda_2 = \frac{-(1+\sigma)+\sqrt{(1+\sigma)^2-4\sigma(1-r)}}{2}$ and $\lambda_3 = \frac{-(1+\sigma)-\sqrt{(1+\sigma)^2-4\sigma(1-r)}}{2}$. Here λ_1 and λ_3 are always negative and λ_2 is negative when $0 < r < 1$ and positive when $r > 1$.
Thus $E_1(0,0,0)$ is stable when $0 < r < 1$ and unstable when $r > 1$. This is independent of the choice of fractional order α.
Thus behavior of the equilibrium point $E_1(0,0,0)$ is similar as integer order dual Lorenz system for all $0 < \alpha < 1$. □

Lemma 2. *For all $0 < \alpha < 1$, the equilibrium points $E_2(\sqrt{b(1-r)}, \sqrt{b(1-r)}, r-1)$ and $E_3(-\sqrt{b(1-r)}, -\sqrt{b(1-r)}, r-1)$ are unstable when $0 < r < 1$.*

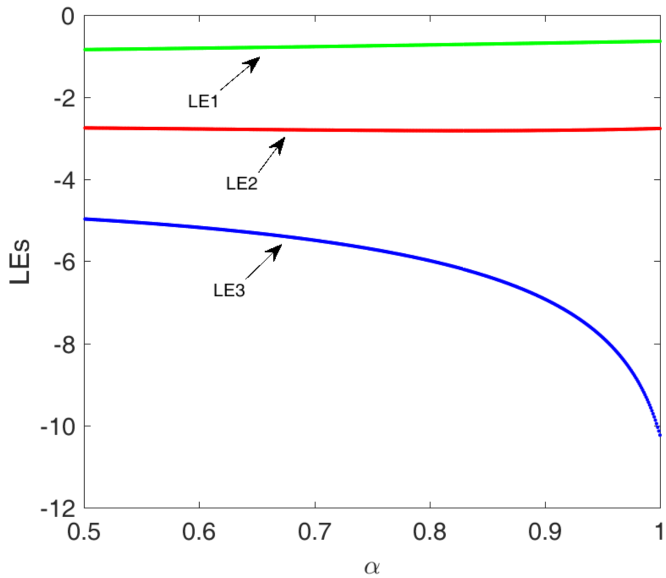


Fig. 3: Plot of the LEs vs α for the FDLS

Proof. The characteristic equation of the Jacobian matrix (3) for equilibrium points E_2 and E_3 with the parameters $\sigma = 10, r = 0.5, b = 8/3$ is given by

$$26.6667 - 28\lambda - 13.6667\lambda^2 - \lambda^3 = 0 \tag{5}$$

The eigenvalues of the Jacobian matrix (3) are $\lambda_1 = -10.8632$, $\lambda_2 = -3.50404$ and $\lambda_3 = 0.700555$. Here λ_1 and λ_2 are negative and λ_3 is positive, therefore, the equilibrium points E_2 and E_3 are unstable.
Similar to the previous lemma, the behaviors of the equilibrium points E_2 and E_3 are similar as integer order dual Lorenz system for all $0 < \alpha < 1$. □

Table 3: Time evolution of the LEs for the FDLS for $\alpha = 0.8$.

t	LE-1	LE-2	LE-3
50.00	-0.6226	-2.7427	-6.1017
100.00	-0.5950	-2.7268	-6.1448
150.00	-0.5859	-2.7216	-6.1592
200.00	-0.5813	-2.7189	-6.1664
250.00	-0.5785	-2.7173	-6.1707
300.00	-0.5767	-2.7163	-6.1736
350.00	-0.5754	-2.7155	-6.1756
400.00	-0.5744	-2.7150	-6.1772
450.00	-0.5736	-2.7145	-6.1784
500.00	-0.5730	-2.7142	-6.1793

4 Numerical simulations and results

In numerical simulations, the parameters of the fractional order dual Lorenz system are considered as $\sigma = 10, r = 0.5, b = 8/3$ and initial value is taken as $[0.1, 0.1, 0.1]$. The Benettin-Wolf algorithm is used for the evaluation of Lyapunov Exponent.
The time evolution of the LEs of the FDLS for different parameters is depicted through figs. 1a to 1c, 2a to 2c and 3 and tables 1 to 3. While tables 1 to 3 and figs. 1a to 1c show the Lyapunov exponent vs time for different fractional orders α, figs. 2a to 2c are shown to depict the Lyapunov exponent vs r for different fractional orders α. fig. 3 is plotted for the Lyapunov exponent vs α. The effort is to find the existence of chaos by varying the parameters using the Lyapunov exponent. As we know that if the Lyapunov exponent is positive, then the system is chaotic.
It is clear from table 1 that the FDLS has Lyapunov exponents $(-0.4854, -2.6761, -9.6809)$ for $\alpha = 0.99$. Similarly, table 2 shows that the FDLS has Lyapunov exponents $(-0.5275, -2.7124, -7.1165)$ for $\alpha = 0.9$. In the same way, table 3 indicates that the FDLS has Lyapunov exponents $(-0.5730, -2.7142, -6.1793)$ for $\alpha = 0.8$.

figs. 2a to 2c present the plots of the Lyapunov exponent versus the parameter r for different values of the fractional-order parameter, while fig. 3 illustrates the variation of the Lyapunov exponent with respect to the fractional-order parameter. The nature of curves for the Lyapunov exponents are similar for the different fractional order derivatives, which show that there is very less effect of the fractional order derivative on the dynamical analysis of the novel FDLS. From the tabulated results and the graphical analysis, it is evident that the Lyapunov exponents remain negative throughout the considered range. This confirms the absence of chaotic behavior in the fractional dual Lorenz system for all parameter values under investigation.

5 Conclusion

The analysis of this article accomplishes two significant objectives. The first study is system's behavior close to its equilibrium points. The investigation of FDLS's dynamic behaviour for different parameters is the second study. For the selected set of parameters, it is found that the fractional order dual Lorenz system does not show the chaotic behaviour. This result is similar to the integer order system already studied (Zlatanovska and Piperevski [6]). The addition of one extra parameter, namely the order of the derivative, makes the study of fractional-order systems particularly interesting. Such systems exhibit rich dynamical behavior and can effectively model many real-world phenomena involving fractional derivatives. The system's behavior is demonstrated through numerical simulations as well as visual presentation based on MATLAB. Future work may focus on the study of complex fractional Lorenz system and its dynamical analysis.

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Declarations

- The authors received no specific funding for this study.
- The authors declare that they have no conflicts of interest to report regarding the present study.
- No Human subject or animals are involved in the research.
- All authors have mutually consented to participate.
- All the authors have consented the Journal to publish this paper.
- Authors declare that all the data being used in the design and production cum layout of the manuscript is declared in the manuscript.

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