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An Anisotropic and Spatially Homogeneous Bianchi type-I Cosmological model in Saez-Ballester Cosmology Filled with Viscous Fluid

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Abstract

This paper presents an exact Bianchi type-I cosmological model in the Saez-Ballester scalar-tensor theory of gravitation in the presence of a viscous fluid. Assuming that the free gravitational field is of Petrov type-D, exact expressions are obtained for the metric potentials, scalar field, and several kinematical parameters. The behaviour of the spatial volume, expansion scalar, mean Hubble parameter, shear scalar, energy density, and isotropic pressure is discussed in detail. In particular, the model exhibits an initial singularity, a finite turnover time controlled by the coefficient of viscosity, and a decay of shear with cosmic time.

Keywords: Saez-Ballester theory of gravitation, Anisotropic, Bianchi type-I, Cosmological model, Petrov type-D, Spatially homogeneous

1 Introduction

Einstein's general theory of relativity accounts for numerous astrophysical phenomena but struggles to explain the late-time acceleration and expansion of the universe. To address these shortcomings, several alternative theories have been proposed, including the Brans-Dicke theory[1, 2], Barber's self-creation theory[3], and the Saez-Ballester theory of gravitation[4].

In this paper, we adopt the Saez-Ballester theory of gravitation as the theoretical framework for our study. This theory modifies the Einstein field equations by introducing a dimensionless scalar field (ϕ) that couples directly with the metric g_{ij}. The physical motivation for choosing this theory lies in its ability to naturally account for the late-time

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accelerated expansion of the universe without requiring the ad hoc inclusion of a cosmological constant or exotic dark energy. The scalar field provides a viable mechanism to explain the missing matter problem in non-flat FRW universes and offers a satisfactory description of the weak-field regime, where general relativity faces significant challenges[4]. Given these advantages, numerous researchers have investigated cosmological models within the framework of the Saez-Ballester theory of gravitation[5–11].

The universe currently exhibits spherical symmetry in its evolution, with an isotropic and homogeneous matter distribution overall. However, due to the unknown types of matter fields in the early cosmos, the universe likely didn't possess such a smooth structure in its initial stages. Additionally, no observational data exists from the period before recombination. Thus, exploring anisotropy in the early universe is natural to address questions like the local anisotropy observed in galaxies, clusters, and superclusters today. The large-scale structure of the current universe is best represented by FRW models due to their homogeneity and isotropy. However, anisotropic background models are more suitable for describing the early stages of the universe's evolution.

Since the 1960s, there has been significant interest in studying Bianchi cosmologies, which play a crucial role in theoretical cosmology by presenting a spatially homogeneous cosmos[12]. Homogeneous cosmological models can be used to examine various aspects of the physical universe. The Bianchi type-I model, a straightforward generalization of the flat FRW model, is one of the simplest models of the universe with an anisotropic background that describes our homogeneous and flat universe. Several researchers have discussed the anisotropic Bianchi type-I model under various conditions and assumptions. Suresh Kumar and C.P. Singh[13, 14] have studied an exact Bianchi type-I cosmological model filled with perfect fluid in the Saez-Ballester theory of gravitation, along with a special law of variation for Hubble's parameter. B. Saha[15], P. Rai et al.[16] have studied the Bianchi type-I cosmological model in the Brans-Dicke theory of gravitation, and R. Bali and Anjali[17] have discussed the Bianchi type-I cosmological model with a magnetized string, among others.

The role of viscosity in cosmological evolution has a long and rich history. Misner[18] first demonstrated that neutrino viscosity in the early universe could damp out large initial anisotropies, thereby explaining the observed isotropy of the cosmos. Collins and Stewart[19] extended this analysis to a qualitative study of dissipative processes in spatially homogeneous Bianchi models. Weinberg[20] further investigated the role of bulk and shear viscosity in entropy generation, showing that bulk viscosity effectively reduces the total pressure and modifies expansion dynamics. The possibility that bulk viscosity can fundamentally alter the singularity structure of big-bang models was explored by Murphy[21]. A comprehensive review of viscous effects in inflationary cosmology was provided by Grøn[22]. More recently, Brevik and Gorbunova[23] have shown that bulk viscosity can mimic dark energy effects and drive accelerated expansion. These findings continue to motivate active research into viscous cosmological models and scalar-tensor anisotropic frameworks in recent years[24–26].

In this paper, we investigate the Bianchi type-I spacetime within the framework of the Saez-Ballester theory of gravitation, incorporating a viscous fluid and imposing the Petrov type-D condition on the free gravitational field. While previous studies have explored Bianchi type-I models with perfect fluids in Saez-Ballester gravity or viscous fluids in other scalar-tensor theories, the novelty of the present work lies in three distinct aspects:

(i) the simultaneous inclusion of viscous dissipation and the Petrov type-D algebraic constraint within the Saez-Ballester framework, a configuration that has not been investigated in existing literature;

(ii) the exact analytical derivation of a viscosity-controlled turnover time $t = 1/(2\eta)$ that separates the expanding and contracting phases, contrasting with works that rely strictly on numerical or qualitative methods; and

(iii) the demonstration that the scalar-field contribution to the energy density and pressure takes the exact factored form $\lambda\, e^{4\eta t}/t^2$, where λ is a genuine constant determined by the integration constants of the scalar-field equation.

A detailed discussion of the effects of viscosity on the kinematic parameters and the physical domain of applicability of the model is provided.

2 Line Element and Field Equations

Here we consider the anisotropic and spatially homogeneous Bianchi type-I metric:

$$ds^2 = -dt^2 + A^2dx^2 + B^2dy^2 + C^2dz^2 \tag{1}$$

where, A=A(t), B=B(t) and C=C(t).
Field equations in Saez-Ballester theory of gravitation are given by:

$$G_{ij} - \omega\phi^n(\phi_{,i}\phi_{,j} - \frac{1}{2}g_{ij}\phi_{,k}\phi^{,k}) = -T_{ij} \tag{2}$$

$$2\phi^n\phi^{,i}_{,i} + n\phi^{n-1}\phi_{,k}\phi^{,k} = 0 \tag{3}$$

In this context, G_{ij} denotes the Einstein tensor, ω signifies the dimensionless coupling constant, and n stands for an arbitrary constant. The symbol T_{ij} represents the energy-momentum tensor, while ϕ is the scalar field. The comma indicates partial differentiation, and the semi-colon denotes partial covariant differentiation with respect to time (t).

The energy conservation equation, which is a direct consequence of equations (2) and (3), is expressed as follows:

$$T^{ij}_{,j} = 0 \tag{4}$$

According to the formulation presented by Landau and Lifshitz[27], when dealing with a distribution of viscous fluid, the stress-energy tensor is expressed as follows:

$$\overline{T}^j_i = (\overline{\epsilon} + \overline{p})\overline{v}_i\overline{v}^j + \overline{p\epsilon_i}^j - \eta(\overline{v}^j_{i;} + \overline{v}^j_{;i} + \overline{v}^j\overline{v}^1\overline{v}_{i;1} + \overline{v}_i\overline{v}^1\overline{v}^j_{;1}) - \left(\xi - \frac{2}{3}\eta\right)\overline{v}^1_{;1}(\overline{g}^j_i + \overline{v}_i\overline{v}^j) \tag{5}$$

together with

$$g_{ij}v^iv^j = -1 \tag{6}$$

In this context, v^i denotes the flow vector, which satisfies equation (6). The term $\overline{p}$ refers to the isotropic pressure within the fluid, while $\overline{\epsilon}$ signifies the energy density of the fluid. Here η denotes the coefficient of shear viscosity and ξ denotes the coefficient of bulk viscosity. The semicolons indicate covariant differentiation.
Here we consider the coordinate system to be co-moving. Hence, $v^i = 0 \ \forall \ i = 1, 2, 3 \ and \ v^4 = 1$.

Consequently, for the spacetime characterized by the metric in (1), the corresponding field equations provided in (2) and (3), along with the stress-energy tensor defined in (5), can be expressed as follows:

$$\frac{B_{44}}{B} + \frac{C_{44}}{C} + \frac{B_4 C_4}{BC} - \frac{\omega}{2}\phi^n {\phi_4}^2 = -\overline{p} + 2\eta\frac{A_4}{A} + \left(\xi - \frac{2}{3}\eta\right)\overline{v}^l_{;l} \tag{7}$$

$$\frac{A_{44}}{A} + \frac{C_{44}}{C} + \frac{A_4 C_4}{AC} - \frac{\omega}{2}\phi^n {\phi_4}^2 = -\overline{p} + 2\eta\frac{B_4}{B} + \left(\xi - \frac{2}{3}\eta\right)\overline{v}^l_{;l} \tag{8}$$

$$\frac{A_{44}}{A} + \frac{B_{44}}{B} + \frac{A_4 B_4}{AB} - \frac{\omega}{2}\phi^n {\phi_4}^2 = -\overline{p} + 2\eta\frac{C_4}{C} + \left(\xi - \frac{2}{3}\eta\right)\overline{v}^l_{;l} \tag{9}$$

$$\frac{A_4 B_4}{AB} + \frac{B_4 C_4}{BC} + \frac{A_4 C_4}{AC} + \frac{\omega}{2}\phi^n {\phi_4}^2 = \overline{\epsilon} \tag{10}$$

$$\phi_{44} + \phi_4\left(\frac{A_4}{A} + \frac{B_4}{B} + \frac{C_4}{C}\right) + \frac{n}{2\phi}{\phi_4}^2 = 0 \tag{11}$$

For the viscous fluid, equation (4) gives the energy-conservation law:

$$\overline{\epsilon}_4 + (\overline{p} + \overline{\epsilon})\left(\frac{A_4}{A} + \frac{B_4}{B} + \frac{C_4}{C}\right) = \xi\left(\overline{v}^l_{;l}\right)^2 + 2\eta\,\overline{\sigma}_{ij}\overline{\sigma}^{ij} \tag{12}$$

where the right-hand side represents viscous entropy production due to bulk and shear viscosity, respectively; it vanishes in the perfect-fluid limit $\eta = \xi = 0$.
Here, the suffix 4 represents ordinary differentiation with respect to time t. Also we take ξ and η to be constant.

Equations (7) through (11) constitute a set of five equations involving the unknown variables A, B, C, ϕ, $\overline{p}$, and $\overline{\epsilon}$. To uniquely determine these six unknowns, an additional equation is needed that establishes a relationship among these variables. To address this requirement, we will introduce a mathematical assumption to provide the necessary additional equation. This assumption is given as:

$$C^{12}_{12} = C^{13}_{13} \tag{13}$$

or,

$$C^{12}_{12} = C^{23}_{23} \tag{14}$$

These conditions establish a relationship among the unknown variables and lead to a Petrov type-D classification, as we now show. For a diagonal Bianchi type-I metric the magnetic part of the Weyl tensor vanishes identically, so the Weyl tensor is purely electric[12, 28]. The electric Weyl tensor $E_{ij} = C_{0i0j}$ is diagonal in the natural orthonormal frame, with eigenvalues e_1, e_2, e_3 satisfying $e_1 + e_2 + e_3 = 0$. The spatial Weyl components for a purely electric Weyl tensor are given by[28]

$$C_{ijkl} = \delta_{ik}E_{jl} - \delta_{il}E_{jk} - \delta_{jk}E_{il} + \delta_{jl}E_{ik}, \tag{15}$$

from which one obtains $C^{12}_{12} = e_1 + e_2 = -e_3$ and $C^{13}_{13} = e_1 + e_3 = -e_2$ (these equalities hold in both the orthonormal and coordinate frames for the diagonal metric). Therefore, the condition $C^{12}_{12} = C^{13}_{13}$ in (13) is equivalent to $e_2 = e_3$, i.e., two eigenvalues of the electric Weyl tensor coincide, which is precisely the criterion for Petrov type-D. Similarly, equation (14) yields $e_1 = e_3$. Since we require an anisotropic model with A, B, and C mutually unequal, these conditions serve as non-trivial differential constraints on the metric functions. In this paper, we proceed with equation (13).

3 The Cosmological Model

Equation (13) gives:

$$\frac{B_{44}}{B} - \frac{C_{44}}{C} + \frac{2A_4}{A}\left(\frac{C_4}{C} - \frac{B_4}{B}\right) = 0 \tag{16}$$

Equations (7) and (8) give

$$\frac{B_{44}}{B} - \frac{A_{44}}{A} + \frac{B_4C_4}{BC} - \frac{A_4C_4}{AC} = 2\eta\left(\frac{A_4}{A} - \frac{B_4}{B}\right) \tag{17}$$

Equations (8) and (9) give

$$\frac{C_{44}}{C} - \frac{B_{44}}{B} + \frac{A_4C_4}{AC} - \frac{A_4B_4}{AB} = 2\eta\left(\frac{B_4}{B} - \frac{C_4}{C}\right) \tag{18}$$

From (16) and (18) we get

$$\frac{3A_4}{A}\left(\frac{C_4}{C} - \frac{B_4}{B}\right) = 2\eta\left(\frac{B_4}{B} - \frac{C_4}{C}\right) \tag{19}$$

Since $B \neq C$, equation (19) gives

$$\frac{3A_4}{A} = -2\eta \tag{20}$$

which on integration gives

$$A = k_1 e^{-\frac{2}{3}\eta t} \tag{21}$$

where, k_1 is an arbitrary integration constant.
Using equation (21) in equation (16) we get:

$$B_4 C - B C_4 = k_2 e^{-\frac{4}{3}\eta t} \tag{22}$$

where k_2 is an arbitrary integration constant.

Using equation (21) in equation (17) we get

$$\frac{B_44}{B} + \frac{B_4 C_4}{BC} + \frac{2}{3}\eta\frac{C_4}{C} + 2\eta\frac{B_4}{B} = -\frac{8}{9}\eta^2 \tag{23}$$

By substituting $\frac{B}{C} = \gamma$ and $BC = \delta$, we obtain $B^2 = \gamma\delta$ and $C^2 = \frac{\delta}{\gamma}$, which reduces equation (22) to

$$\frac{\gamma_4 \delta}{\gamma} = k_2 e^{-\frac{4}{3}\eta t} \tag{24}$$

and from equation (23) we get

$$9\delta_{44} + 24\eta\delta_4 + 16\eta^2\delta = 0 \tag{25}$$

which gives

$$\delta = (k_3 + k_4 t)e^{-\frac{4}{3}\eta t} \tag{26}$$

where k_3 and k_4 are arbitrary integration constants.
Using equation (24) and equation (26) we get

$$\gamma = (k_3 + k_4 t)^k \tag{27}$$

where k is a constant.
Thus, equations (26) and (27) give

$$B^2 = (k_3 + k_4 t)^{(1+k)} e^{-\frac{4}{3}\eta t} \tag{28}$$

and

$$C^2 = (k_3 + k_4 t)^{(1-k)} e^{-\frac{4}{3}\eta t} \tag{29}$$

Thus with viscous fluid distribution the metric (1) becomes:

$$ds^2 = -dt^2 + {k_1}^2 e^{-\frac{4}{3}\eta t} dx^2 + (k_3 + k_4 t)^{(1+k)} e^{-\frac{4}{3}\eta t} dy^2 + (k_3 + k_4 t)^{(1-k)} e^{-\frac{4}{3}\eta t} dz^2 \tag{30}$$

For a representative subclass of the above family, we choose the integration constants $k_1 = 1$, $k_3 = 0$, and $k_4 = 1$. Then the metric (30) reduces to:

$$ds^2 = -dt^2 + e^{-\frac{4}{3}\eta t}dx^2 + t^{(1+k)}e^{-\frac{4}{3}\eta t}dy^2 + t^{(1-k)}e^{-\frac{4}{3}\eta t}dz^2 \tag{31}$$

Some Kinematical Properties

The energy density and isotropic pressure for the metric (31) are obtained as:

$$\bar{\epsilon} = \frac{1-k^2}{4t^2} - \frac{4\eta}{3t} + \frac{4}{3}\eta^2 + \frac{\omega}{2}\phi^n{\phi_4}^2 \tag{32}$$

$$\overline{p} = \frac{1-k^2}{4t^2} + \frac{\omega}{2}\phi^n{\phi_4}^2 + \frac{1}{t}\left(\frac{4\eta}{3} + \xi\right) - \frac{4}{3}\eta^2 - 2\xi\eta \tag{33}$$

It can be verified that the expressions (32) and (33) satisfy the viscous energy conservation equation (12) identically, with the right-hand side evaluating to $\xi\theta^2 + 4\eta\sigma^2$, confirming the self-consistency of the solution. Also for (31) the scalar field(ϕ) is obtained as:

$$\phi = c_2\left[2c_1 + (2+n)\mathrm{Ei}(2\eta t)\right]^{\frac{2}{2+n}} \tag{34}$$

Here, c_1 and c_2 are arbitrary integration constants, and we assume $n \neq -2$ throughout. This form follows by introducing $Y = \phi^{\frac{n+2}{2}}$ (which requires $n \neq -2$), reducing equation (11) to

$$Y_{44} + \left(\frac{1}{t} - 2\eta\right)Y_4 = 0. \tag{35}$$

Integrating once gives $Y_4 = C_1\, e^{2\eta t}/t$, where C_1 is a constant. Since $\phi_4 = \frac{2}{n+2}\,\phi^{-n/2}\,Y_4$, it follows that

$$\frac{\omega}{2}\phi^n\phi_4^2 = \frac{2\omega C_1^2}{(n+2)^2}\,\frac{e^{4\eta t}}{t^2} \;=\; \lambda\,\frac{e^{4\eta t}}{t^2}, \tag{36}$$

where $\lambda = 2\omega C_1^2/(n+2)^2$ is a genuine constant of the solution, which is determined entirely by the coupling constant ω and the integration constant C_1. This factorisation is exact and holds for all $n \neq -2$. The exceptional case $n = -2$, in which the substitution $Y = \phi^{(n+2)/2}$ degenerates, would require a separate treatment and is not considered here.
For the model (31), the volume element is given by:

$$V = te^{-2\eta t} \tag{37}$$

For the model (31), the flow vector v^i of the fluid distribution is given as

$$v^i = 0 \;\forall\; i = 1, 2, 3$$

$$v^4 = 1 \tag{38}$$

The flow is geodetic as $v^i_{;j}v^j = 0$. Also the rotation tensor vanishes as $W_{ij} = v_{i;j} - v_{j;i} = 0$. For the flow vector v^i, the scalar expansion is obtained as:

$$\theta = \left(\frac{1}{t} - 2\eta\right) \tag{39}$$

From the above expression it follows that viscosity retards the model's expansion. In particular, $\theta > 0$ for $t < \frac{1}{2\eta}$, $\theta = 0$ at $t = \frac{1}{2\eta}$, and $\theta < 0$ for $t > \frac{1}{2\eta}$. The Hubble's parameter is obtained as:

$$H = \frac{\theta}{3} = \frac{1}{3}\left(\frac{1}{t} - 2\eta\right) \tag{40}$$

The components of shear tensor which are non- vanishing are given by

$$\begin{aligned}
\sigma_{11} &= -\frac{1}{3t}e^{-\frac{4}{3}\eta t} \\
\sigma_{22} &= \frac{1+3k}{6}t^k e^{-\frac{4}{3}\eta t} \\
\sigma_{33} &= \frac{1-3k}{6}t^{-k} e^{-\frac{4}{3}\eta t} \\
\sigma_{44} &= 0
\end{aligned} \tag{41}$$

The shear σ is given by:

$$\sigma^2 = \frac{1+3k^2}{12t^2} \tag{42}$$

The shear scalar decays monotonically as t^{-2}. However, the ratio of the shear scalar to the expansion scalar is

$$\frac{\sigma^2}{\theta^2} = \frac{1+3k^2}{12(1-2\eta t)^2}, \tag{43}$$

which increases during the expanding phase and diverges as $t \to 1/(2\eta)$, where the expansion vanishes. Thus, although the absolute shear decreases, the anisotropy becomes relatively more significant as the expansion slows toward the turnover. In the contracting regime $(t > 1/(2\eta))$, $\sigma^2 \to 0$ and $\sigma^2/\theta^2 \to (1+3k^2)/(48\eta^2 t^2) \to 0$ as $t \to \infty$.

4 Numerical Analysis

To illustrate the behaviour of the model, we consider a representative parameter set

$$\eta = 0.1, \qquad k = 0.5, \qquad \xi = 0.03.$$

As shown in equation (36), the scalar-field contribution is exactly $\lambda\, e^{4\eta t}/t^2$, where $\lambda = 2\omega C_1^2/(n+2)^2$ is a constant of the solution. We take $\lambda = 0.02$ for the numerical plots below. The density and pressure then take the form

$$\bar{\epsilon} = \frac{1-k^2}{4t^2} - \frac{4\eta}{3t} + \frac{4}{3}\eta^2 + \lambda\frac{e^{4\eta t}}{t^2}, \tag{44}$$

$$\bar{p} = \frac{1-k^2}{4t^2} + \lambda\frac{e^{4\eta t}}{t^2} + \frac{1}{t}\left(\frac{4\eta}{3} + \xi\right) - \frac{4}{3}\eta^2 - 2\xi\eta. \tag{45}$$

Table 1: Representative numerical values for the subclass (31) with $\eta = 0.1$, $k = 0.5$, $\xi = 0.03$, and $\lambda = 0.02$. The turnover time is $t = 1/(2\eta) = 5$.

t	V	θ	H	σ^2	$\bar{\epsilon}$	$\bar{p}$
1	0.8187	0.8000	0.2667	0.1458	0.0973	0.3613
2	1.3406	0.3000	0.1000	0.0365	0.0047	0.1203
5	1.8394	0.0000	0.0000	0.0058	0.0001	0.0267
10	1.3534	−0.1000	−0.0333	0.0015	0.0128	0.0098

Table 1 shows that the spatial volume increases initially, reaches its maximum near $t = \frac{1}{2\eta} = 5$, and then decreases. The scalar expansion and the mean Hubble parameter are positive before the turnover time and become negative afterwards, while the shear scalar remains positive and decays as t^{-2}.

The numerical profiles confirm the analytical structure of the solution. In particular, viscosity introduces a finite turnover time $t = 1/(2\eta) = 5$ in the expansion, while the shear scalar decreases monotonically as t^{-2}. During the expanding phase ($t < 5$), the energy density and pressure are dominated by the geometric t^{-2} terms and decrease rapidly from their singular values. After the turnover, the scalar-field contribution $\lambda\, e^{4\eta t}/t^2$ grows exponentially and eventually dominates, as is evident from the increase of $\bar{\epsilon}$ between $t = 5$ and $t = 10$ in Table 1. Consequently, the model is physically well-behaved primarily in the expanding phase $0 < t < 1/(2\eta)$, and the contracting phase should be regarded as a mathematical feature of the solution rather than a physically realistic epoch.

5 Discussion

This paper presents an anisotropic and spatially homogeneous Bianchi type-I cosmological model in the framework of the Saez-Ballester scalar-tensor theory of gravitation[4], filled with a viscous fluid and restricted by the Petrov type-D condition. The Petrov type-D assumption has been justified through the eigenvalue structure of the Weyl tensor for diagonal Bianchi type-I metrics[12, 28]. Exact solutions of the field equations have been obtained for the metric potentials and the scalar field, and the corresponding kinematical and physical quantities have been discussed for the subclass described by (31).

For this model, the spatial volume is given by $V = te^{-2\eta t}$, the scalar expansion is $\theta = \frac{1}{t} - 2\eta$, and the mean Hubble parameter is $H = \frac{1}{3}\left(\frac{1}{t} - 2\eta\right)$. Hence, as $t \to 0$, the model starts from an initial singular state with vanishing spatial volume and divergent expansion. This singular behaviour is consistent with the classical big-bang scenario; as Murphy[21] demonstrated, bulk viscosity can in principle push such singularities to the infinite past, suggesting that the inclusion of viscous effects fundamentally alters the singularity structure. The presence of viscosity significantly affects the dynamics: the universe expands for

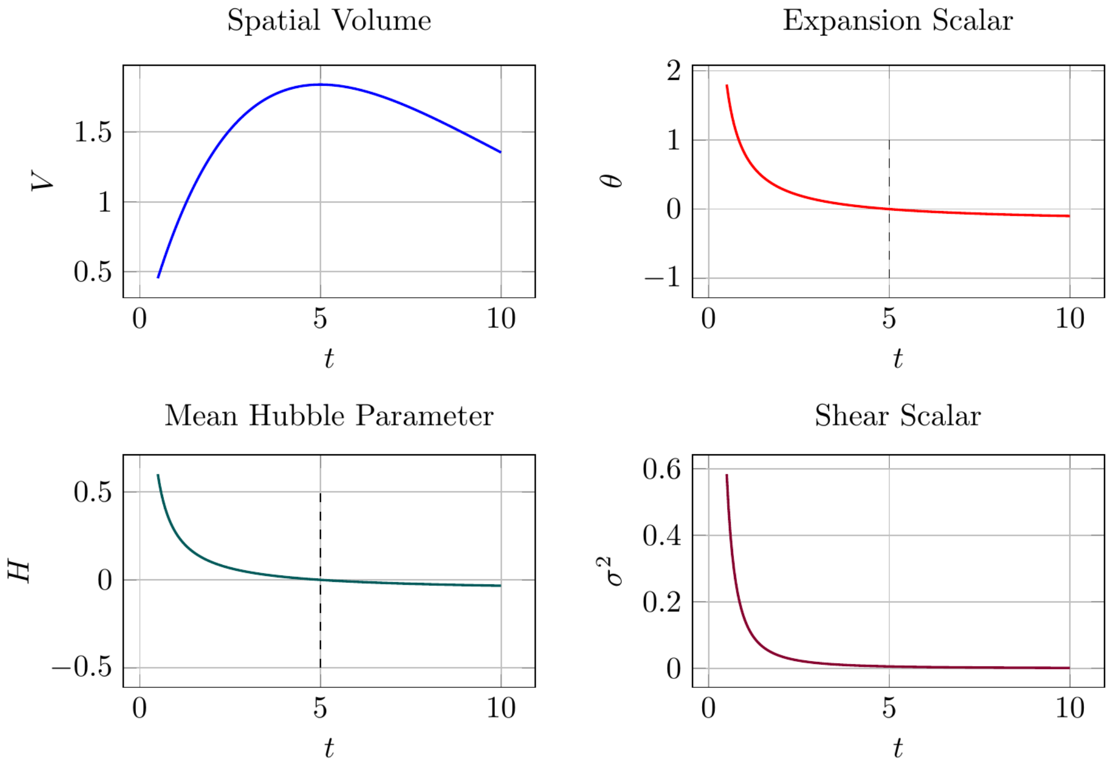

Fig. 1: Evolution of the principal kinematical quantities for $\eta = 0.1$ and $k = 0.5$. The dashed vertical line marks the turnover time $t = \frac{1}{2\eta} = 5$.

$t < \frac{1}{2\eta}$, attains its maximum volume at $t = \frac{1}{2\eta}$, and contracts for $t > \frac{1}{2\eta}$. Thus the coefficient of viscosity controls the turnover time of the model. This finding is in agreement with the general result of Weinberg[20], who showed that bulk viscosity acts as an effective reduction in the total pressure and consequently modifies the expansion history.

A key result of this work is that the scalar-field contribution to the energy density and pressure takes the exact form $\lambda\, e^{4\eta t}/t^2$, where $\lambda = 2\omega C_1^2/(n+2)^2$ is a genuine constant of the solution (see equation (36)). This factorisation is not an approximation but follows directly from the integrability of the scalar-field equation. During the expanding phase ($0 < t < 1/(2\eta)$), the geometric terms of order t^{-2} dominate and the energy density decreases rapidly from its singular initial value. However, after the turnover time, the exponentially growing scalar-field term eventually dominates, causing $\bar{\epsilon}$ to increase, as confirmed by the numerical data in Table 1. The model is therefore physically applicable primarily to the expanding phase; the contracting phase ($t > 1/(2\eta)$) represents a mathematical regime of the exact solution in which the energy conditions require separate analysis.

The shear scalar is found to be $\sigma^2 = \frac{1+3k^2}{12t^2}$, which decays monotonically as t^{-2} independently of the viscosity coefficient η. Notably, this decay is a purely kinematic feature: the shear decreases whether the model is expanding or contracting. However, the anisotropy ratio $\sigma^2/\theta^2 = (1+3k^2)/[12(1-2\eta t)^2]$ increases during the expanding phase and diverges at the turnover time $t = 1/(2\eta)$, where $\theta = 0$. This means that while the absolute anisotropy diminishes, it becomes relatively more prominent as the expansion stalls. The divergence of

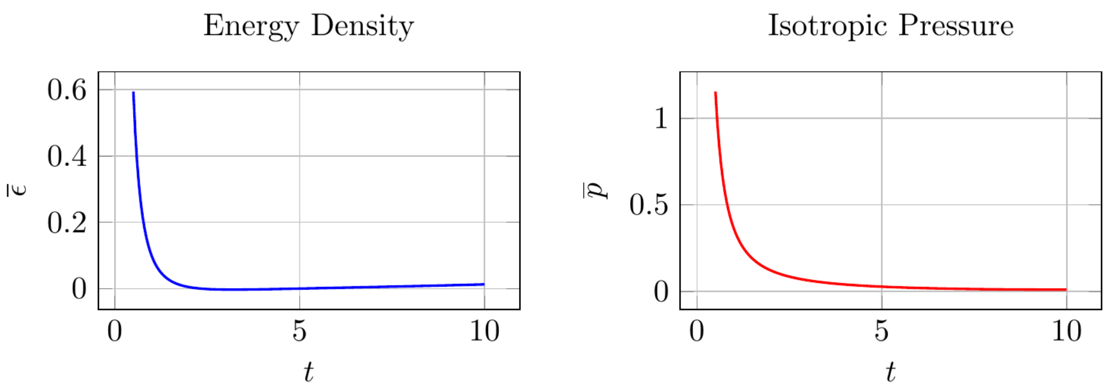

Fig. 2: Numerical behaviour of the energy density and isotropic pressure for $\eta = 0.1$, $k = 0.5$, $\xi = 0.03$, and $\lambda = 0.02$.

σ^2/θ^2 is a direct consequence of θ passing through zero at the turnover; it does not signal a physical instability but rather reflects the unsuitability of σ^2/θ^2 as an anisotropy measure near the turnover where $\theta = 0$.

The role of viscous dissipation in damping primordial anisotropies was first established by Misner[18] and further analyzed by Collins and Stewart[19] for general Bianchi cosmologies. From the observational perspective, the Planck satellite data[29] places stringent constraints on the present-day expansion anisotropy ($\Delta T/T \sim 10^{-5}$). Since the present model describes the early expanding phase of the universe rather than the late-time epoch probed by the CMB, a direct comparison with Planck constraints is not pursued; the observational data is noted here only to motivate the physical relevance of studying anisotropy dissipation mechanisms.

6 Conclusion

In summary, the principal features of the model are: (i) an initial big-bang singularity; (ii) a finite expanding phase whose duration is controlled by the viscosity coefficient; (iii) monotonic decay of the absolute shear scalar as t^{-2}, with the relative anisotropy measure σ^2/θ^2 increasing toward the turnover; and (iv) an exact, closed-form expression for the scalar-field contribution to the matter sector, valid for all $n \neq -2$. These features make the solution a useful exact model for studying the interplay of viscous effects, scalar fields, and anisotropy in the Saez-Ballester framework.

Declarations

- The authors received no specific funding for this study.

- The authors declare that they have no conflicts of interest to report regarding the present study.
- No Human subject or animals are involved in the research.
- All authors have mutually consented to participate.
- All the authors have consented the Journal to publish this paper.
- Authors declare that all the data being used in the design and production cum layout of the manuscript is declared in the manuscript.

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