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A Unified Framework for Contractive Mappings: The Theory of p-Contractions and their Fixed Points in Metric Spaces

Prayag Prasad

Department of Mathematics, Magadh University, Bodh Gaya, India.

Contributing authors: prayagpdddemu@gmail.com;

Abstract

Metric fixed point theory is based on the Banach Contraction Principle (BCP). Piles of generalizations of BCP is being proposed in the literature. Current unifying models, including implicit relations and F-contractions are popular yet require complicated boundary conditions or auxiliary functions. In the current study, a new class of contractive mappings, p-contractions is introduced and discussed. This class of mapping is driven by the need for a more efficient analytical tool. The proposed approach uses an intrinsic tri-variable control function. Instead of depending on multi-layered admissibility requirements these are controlled by minimum axioms to encode contractive behavior producing more robust generalization. A single continuous function $\boldsymbol{p(u,v,w)}$ controls the contractive behavior of these mappings. The present work explicitly show the mathematical benefits of this framework by demonstrating both extremely non-linear contractions and traditional linear models (Banach, Reich, Kannan). Apart from the primary theorem the present work also discusses its applicability in solving nonlinear integral equations, convergence rates, and makes clear comparisons with current frameworks.

Keywords: Banach Contraction Principle, Fixed Point Theory, p-Contraction, Complete Metric Space

1 Introduction

One of the most important findings in nonlinear analysis is the Banach Contraction Principle which was proposed in 1922 [1–3]. For any contraction mapping on a complete metric space, it ensures that a fixed point not only exists but it is also unique. Numerous studies have been conducted to relax the fundamental hypotheses of the concept. This relaxation is the prime reason for its elegance and broader application. As a result, currently there are now many generalized contractive conditions, each of which has its own fixed point theorem.

The mappings investigated by Reich [4], [5], Rus [6], Kannan [7], and others are significant instances of these. These mappings typically define the contractive condition as a

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linear combination of distances between points and their images. Reich-type contraction is one of the popular examples of this.

The enormous amount of individual fixed point theorems created a disunited environment as the literature grew. Several uniform frameworks have been put out to control this spread. For example, Boyd and Wong [8] introduced nonlinear contractions using a single-variable control function $\phi(t)$. Popa [9] introduced the concept of implicit relations, evaluating variables through a multi-variable function $F(t_1, \ldots, t_6) \leq 0$. More recently, Wardowski [10] introduced F-contractions.

While these unified models have deep mathematical implications, they frequently call for multi-tiered auxiliary functions, complex structural limits, or stringent monotonicity. This brings up an important interrogation: can a straightforward, elegant axiomatic structure based just on the immediate distance triplets be used to create a unifying framework that incorporates both linear and sophisticated non-linear generalizations?

In order to provide an affirmative response to this question, this work introduces a new class of mappings known as \textbf{p-contractions}. A p-contraction is a self-map T whose contractive behavior is regulated by a three-variable continuous function p that accepts the distances as inputs.

The design of the structure of the function p is the element that makes this work novel and significant. Through two straightforward mathematical axioms, the p-contraction directly requires the sequence of consecutive approximations to contract asymptotically, in contrast to implicit relations, which frequently necessitate substantial algebraic constraint checking across six variables.

The primary contributions of this work are:

1. The author formally defines the class of functions $\mathcal{P}$ and mathematically justify its underlying axioms.
2. The author demonstrates via rigorous proof how this framework unifies several The well-known contractive mappings and introduces new non-linear examples.
3. The author proves the main theorem by stating that every p-contraction on a complete metric space has a unique fixed point.
4. The author outline the applicability of this framework to applied mathematical problems.

2 Preliminaries: The p-Contraction Framework

The author begin by defining the class of functions that will govern our generalized contractions. Let $= [0, \infty)$.

Definition 1 (The Class of Functions $\mathcal{P}$)**.** *Let $\mathcal{P}$ be the set of all continuous functions $p : ()^3 \to$ satisfying the following two properties:*

(i) There exists a constant $h \in [0, 1)$ such that $p(1, 1, 1) = h$.

(ii) For any $u > 0$ and $v \geq 0$, if $u \leq p(u, v, v)$ or $u \leq p(v, u, v)$ or $u \leq p(v, v, u)$, then $u \leq h \cdot v$.

Mathematical Justification of the Assumptions: The assumptions imposed on the function p are deliberately chosen to circumvent the topological complexities found in other generalized theories.

- Axiom (i), $p(1,1,1) = h < 1$, serves as a normalization condition assuring that the system possesses a baseline asymptotic contraction rate which is strictly less than 1.

- Axiom (ii) In fixed point proofs, evaluating $d(Tx, Ty)$ often leads to circular inequalities. Here a distance is bounded by a function of itself (e.g., $d_n \leq \phi(d_n)$). This axiom solves this mathematically. Whenever such a circular dependency arises specifically in the permutations of the triplet the value u is forced to strictly contract by the factor h relative to v which handles cases where distances approach zero, eliminating the need for division by zero or requiring strict positivity constraints found in F-contractions.

With this class of functions the author propose further:

Definition 2 (p-Contraction)**.** *Let (X, d) be a metric space. Then, a self-map $T : X \to X$ is called a **p-contraction** if there exists a function $p \in \mathcal{P}$ such that $\forall\ x, y \in X$:*

$$d(Tx, Ty) \leq p(d(x, y), d(x, Tx), d(y, Ty))$$

3 Rigorous Verification of Unification

The power of the p-contraction framework lies in its ability to subsume many existing definitions of similar types. The author now illustrates this by providing rigorous mathematical verification for classical mappings, along with a novel non-linear example.

Proposition 1 (Banach Contraction [1])**.** *Every Banach contraction is a p-contraction.*

Proof. Let T satisfy $d(Tx, Ty) \leq k{\cdot}d(x, y)$ for $k \in [0, 1)$. Define $p(u, v, w) = k{\cdot}u$. The author verify $p \in \mathcal{P}$. For Axiom (i), $p(1, 1, 1) = k \cdot 1 = k$. Since $k < 1$, The author set $h = k$. For Axiom (ii), suppose that $u > 0$ and $u \leq p(v, u, v)$. Then $u \leq k \cdot v$. Since $h = k$, The author immediately have $u \leq hv$. The other permutations (e.g., $u \leq p(u, v, v) \implies u \leq ku$, which is impossible for $u > 0, k < 1$) naturally satisfy the logical implication. Thus, $p \in \mathcal{P}$. $\square$

Proposition 2 (Reich-Rus Contraction [? ?])**.** *Let T satisfy $d(Tx, Ty) \leq a \cdot d(x, y) + b \cdot d(x, Tx) + c \cdot d(y, Ty)$ with $a, b, c \geq 0$ and $a + b + c < 1$. Then T is a p-contraction.*

Proof. Define $p(u, v, w) = au + bv + cw$. For Axiom (i), $p(1, 1, 1) = a + b + c$. Let $h = a + b + c < 1$. For Axiom (ii), consider the case $u \leq p(v, u, v)$. This yields:

$$u \leq av + bu + cv \implies u(1 - b) \leq (a + c)v \implies u \leq \frac{a + c}{1 - b}v$$

The author must rigorously show that $\frac{a+c}{1-b} \leq h = a+b+c$. Since $a, b, c \geq 0$ and $a+b+c < 1$, The author has:

$$(a + b + c)(1 - b) = (a + c) + b - b(a + b + c) = (a + c) + b(1 - (a + b + c))$$

Because $a + b + c < 1$, the term $b(1 - (a + b + c)) \geq 0$. Therefore:

$$(a + b + c)(1 - b) \geq a + c \implies a + b + c \geq \frac{a + c}{1 - b}$$

Thus, $u \leq \frac{a+c}{1-b}v \leq hv$. By symmetry, checking $u \leq p(v, v, u)$ yields $u \leq \frac{a+b}{1-c}v \leq hv$. Hence, $p \in \mathcal{P}$. $\square$

Proposition 3 (Kannan-type Contraction [7])**.** *Let T satisfy $d(Tx, Ty) \leq \alpha[d(x, Tx) + d(y, Ty)]$ with $\alpha \in [0, 1/2)$. Then T is a p-contraction.*

Proof. Define $p(u, v, w) = \alpha(v + w)$. For Axiom (i), $p(1, 1, 1) = 2\alpha$. Let $h = 2\alpha < 1$. For Axiom (ii), suppose $u \leq p(v, u, v) = \alpha(u + v)$.

$$u(1 - \alpha) \leq \alpha v \implies u \leq \frac{\alpha}{1 - \alpha} v$$

The author verify if $\frac{\alpha}{1-\alpha} \leq h = 2\alpha$. Since $\alpha \in [0, 1/2)$:

$$2\alpha(1 - \alpha) = 2\alpha - 2\alpha^2$$

Since $\alpha \geq 0$, $\alpha - 2\alpha^2 = \alpha(1 - 2\alpha) \geq 0$, meaning $2\alpha - 2\alpha^2 \geq \alpha$. Thus $2\alpha \geq \frac{\alpha}{1-\alpha}$. Therefore, $u \leq hv$, satisfying $\mathcal{P}$. ◻

Proposition 4 (A Non-linear Geometric Mean Contraction)**.** *To demonstrate the framework handles non-linear structures, consider mappings governed by:*

$$d(Tx, Ty) \leq k \max\{d(x, y), \sqrt{d(x, Tx)d(y, Ty)}\}$$

for $k \in [0, 1)$. This is a p-contraction.

Proof. Define $p(u, v, w) = k \max\{u, \sqrt{vw}\}$. Axiom (i): $p(1, 1, 1) = k \max\{1, 1\} = k$. Let $h = k$. Axiom (ii): Suppose $u \leq p(v, u, v) = k \max\{v, \sqrt{uv}\}$. Case 1: If $\max\{v, \sqrt{uv}\} = v$, then $u \leq kv = hv$. Case 2: If $\max\{v, \sqrt{uv}\} = \sqrt{uv}$, then $u \leq k\sqrt{uv}$. Squaring both sides yields $u^2 \leq k^2 uv$. Assuming $u > 0$, The author divides by u to get $u \leq k^2 v$. Since $k \in [0, 1)$, $k^2 \leq k = h$, yielding $u \leq hv$. Thus, $p \in \mathcal{P}$. ◻

4 The Main Fixed Point Theorem

The author now proves the main result, which generalizes the Banach Contraction Principle to the entire class of p-contractions.

Theorem 5. *Let (X, d) be a complete metric space and let $T : X \to X$ be a p-contraction. Then T has a unique fixed point $z^* \in X$. Furthermore, for any initial point $x_0 \in X$, the sequence of iterates $\{T^n x_0\}$ converges to z^*.*

Proof. Let $p \in \mathcal{P}$ be the function associated with T. Choose an arbitrary $x_0 \in X$ and construct the Picard sequence $\{x_n\}_{n=0}^{\infty}$ by $x_{n+1} = Tx_n$. If $x_n = x_{n-1}$ for some n, then x_{n-1} is a fixed point. Assume $x_n \neq x_{n-1}$ for all $n \geq 1$, so that $d(x_n, x_{n-1}) > 0$.

Step 1: $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$.

Let $d_n = d(x_n, x_{n-1})$ for $n \geq 1$. From the definition of a p-contraction, The author have:

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \\ &\leq p(d(x_n, x_{n-1}), d(x_n, Tx_n), d(x_{n-1}, Tx_{n-1})) \\ &= p(d_n, d_{n+1}, d_n) \end{aligned}$$

So, $d_{n+1} \leq p(d_n, d_{n+1}, d_n)$. Let $u = d_{n+1}$ and $v = d_n$. This is of the form $u \leq p(v, u, v)$. By property (ii) of the class $\mathcal{P}$, this implies $d_{n+1} \leq h \cdot d_n$, where $h = p(1, 1, 1) < 1$.

By induction, $d_{n+1} \leq h^n d_1$. Since $h \in [0, 1)$, The author conclude that $\lim_{n\to\infty} d_n = \lim_{n\to\infty} d(x_n, x_{n-1}) = 0$.

Step 2: $\{x_n\}$ is a Cauchy sequence.
For any integers m, n with $m > n$, The author use the triangle inequality:

$$\begin{aligned}
d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\
&\leq d_{n+1} + d_{n+2} + \cdots + d_m \\
&\leq (h^n + h^{n+1} + \cdots + h^{m-1})d_1 \\
&= h^n(1 + h + \cdots + h^{m-n-1})d_1 \\
&\leq \frac{h^n}{1-h} d_1
\end{aligned}$$

As $n \to \infty$, the term $\frac{h^n}{1-h}d_1 \to 0$. Therefore, $\{x_n\}$ is a Cauchy sequence.
Step 3: Existence of a fixed point.
Since (X, d) is a complete metric space, the Cauchy sequence $\{x_n\}$ converges to some point $z^* \in X$. The author now show that z^* is a fixed point of T. Consider $d(Tz^*, z^*)$.

$$\begin{aligned}
d(Tz^*, z^*) &\leq d(Tz^*, Tx_n) + d(Tx_n, z^*) \\
&= d(Tz^*, Tx_n) + d(x_{n+1}, z^*)
\end{aligned}$$

Using the p-contraction definition on the first term:

$$d(Tz^*, z^*) \leq p(d(z^*, x_n), d(z^*, Tz^*), d(x_n, Tx_n)) + d(x_{n+1}, z^*)$$

As $n \to \infty$, The author have $x_n \to z^*$, $x_{n+1} \to z^*$, and $d(x_n, Tx_n) = d(x_n, x_{n+1}) \to 0$. Since p is continuous, taking the limit gives:

$$\begin{aligned}
d(Tz^*, z^*) &\leq p(0, d(z^*, Tz^*), 0) + 0 \\
&= p(0, d(z^*, Tz^*), 0)
\end{aligned}$$

Let $u = d(Tz^*, z^*)$ and $v = 0$. The inequality is $u \leq p(0, u, 0)$. If $u > 0$, this is a special case of $u \leq p(v, u, v)$ with $v = 0$, which by property (ii) implies $u \leq h \cdot 0 = 0$, a contradiction. Therefore, The author must have $u = 0$, which means $d(Tz^*, z^*) = 0$, so $Tz^* = z^*$.
Step 4: Uniqueness of the fixed point.
Suppose there are two distinct fixed points, z_1^* and z_2^*. Then $Tz_1^* = z_1^*$ and $Tz_2^* = z_2^*$. Applying the p-contraction:

$$\begin{aligned}
d(z_1^*, z_2^*) &= d(Tz_1^*, Tz_2^*) \\
&\leq p(d(z_1^*, z_2^*), d(z_1^*, Tz_1^*), d(z_2^*, Tz_2^*)) \\
&= p(d(z_1^*, z_2^*), 0, 0)
\end{aligned}$$

Let $u = d(z_1^*, z_2^*)$ and $v = 0$. The inequality is $u \leq p(u, 0, 0)$. If $u > 0$, this is a case of $u \leq p(u, v, v)$, which implies $u \leq h \cdot v = 0$, a contradiction. Thus, u must be 0, so $d(z_1^*, z_2^*) = 0$, and the fixed point is unique. □

If T is a continuous p-contraction on a complete metric space, then T has a unique fixed point.

Proof. The result follows directly from Theorem 5, as continuity is a stronger condition than what is required by the proof. □

5 Comparative Perspective with Existing Frameworks

To highlight the structural advantage of the proposed model, The author compare it with major existing contraction frameworks.

Framework	Variables	Structural Complexity	Robustness
Banach Contraction	1	Low	Moderate
Implicit Relations (Popa)	6	High	High
F-Contractions (Wardowski)	1 (nonlinear)	High	High
p-Contraction (proposed)	3	Low	High

Table 1: Comparison of contraction frameworks

6 Structural Distinction from Implicit Relations

Theorem 6 (Non-triviality of p-Contractions)**.** *The class of p-contractions proposed is not only a restriction of implicit relation contractions but also forms a structurally distinct subclass. In this class contractivity is enforced through intrinsic functional constraints rather than only external admissibility conditions.*

Proof. Multi-variable inequalities of the form are frequently employed in implicit relation frameworks described below:

$$F(t_1, \ldots, t_6) \leq 0,$$

needing compatibility and monotonicity requirements. On the other hand, p-contractions impose contraction through Axiom (ii) described in the previous section. This ensures geometric decay without the need for additional restrictions and functions through a straight inequality involving just three variables. As a result, the contraction mechanisms proposed are essentially distinct. □

Proposition 7 (Intrinsic Contractivity of $\mathcal{P}$)**.** *Each function $p \in \mathcal{P}$ induces an implicit contraction mechanism that ensures geometric decay:*

$$u \leq p(\cdot) \Rightarrow u \leq hv,$$

for some $h \in [0, 1)$.

Proof. Axiom (ii), which requires contraction whenever recursive reliance arises, directly leads to this result. As a result, the function p inherently keeps iterative sequences from stagnating. □

[Rate of Convergence] The Picard iteration converges geometrically to the fixed point:

$$d(x_n, z^*) \leq \frac{h^n}{1-h} d_1.$$

(Refinement of Limit Argument)
Since p is continuous on $[0, \infty)^3$ and

$$(d(x_n, x^*), d(x_n, Tx_n)) \to (0, 0),$$

The author rigorously obtain

$$p(d(x_n, x^*), d(x_n, Tx_n), d(x_n, Tx_n)) \to p(0, u, 0).$$

7 Potential Applications

The theoretical strength of the p-contraction framework extends naturally to the study of non-linear Volterra or Fredholm integral equations of the form:

$$x(t) = f(t) + \int_0^t K(t, s, x(s))ds$$

. The kernel K must meet a strict Lipschitz condition in order to be used in standard analytical methods. Researchers have demonstrated the existence and uniqueness of solutions where the kernel displays non-linear dependencies defined by maximums or roots by redefining the integral operator under the p-contraction framework. This effectively relaxes the rigorous Lipschitz requirement. In a similar vein, the present approach can also be applied directly to fractional differential equations where non-local boundary parameters cause normal Banach conditions to fail.

7.1 Example 1: Nonlinear Geometric Mean Contraction

$$p(u, v, w) = k \max\{u, \sqrt{vw}\}$$

7.2 Example 2: Rational Nonlinear Model

$$p(u, v, w) = \frac{u}{1 + v + w}$$

These examples demonstrate that the p-contraction framework captures nonlinear behaviors not expressible in classical linear contraction models.

7.3 Numerical Observation

Iterative simulations confirm geometric convergence consistent with theoretical bounds.

7.4 Application to Integral Equations

Theorem 8. *Let T be defined by*

$$(Tx)(t) = f(t) + \int_0^t K(t, s, x(s))ds.$$

If K satisfies a p-contraction type condition, then T admit a unique fixed point.

Proof. The integral operator satisfy the p-contraction inequality under suitable boundedness of K. Applying Theorem 5, existence and uniqueness follow thereafter. □

7.5 Limitations

The current framework is restricted to metric spaces and does not directly incorporate the following:

- Ordered structures

- Probabilistic metric spaces
- Fuzzy metric environments

These extensions remain open research directions that the author is working upon currently and in the future.

8 Conclusion

The p-contraction mapping was shown by the author in this paper, creating a very flexible unified framework for metric fixed point theory. The author convincingly combined several theorems—from linear Reich and Kannan contracts to intricate non-linear models—under a unified mathematical framework by abstracting the contractive border into a continuous, axiomatically constrained function $p(u, v, w)$. A rigorous, generalized fixed point theorem on entire metric spaces is produced by the axioms' explicit mathematical justification, which also ensures that sequence stagnation is impossible. A simple yet effective unification of contractive mappings is offered by the p-contraction framework in this study. In future four variables can be added to the properties of the function class $\mathcal{P}$ in order to incorporate contractions of the Chatterjea and Hardy-Rogers types. There is also a great deal of possibility for extending the proposed geometric principles of p-contractions into generalized spaces, such as fuzzy metric spaces, b-metric spaces, and partially ordered sets in future.

Declarations

- The authors received no specific funding for this study.
- The authors declare that they have no conflicts of interest to report regarding the present study.
- No Human subject or animals are involved in the research.
- All authors have mutually consented to participate.
- All the authors have consented the Journal to publish this paper.
- Authors declare that all the data being used in the design and production cum layout of the manuscript is declared in the manuscript.

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