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On the Solution of Burgers' Equation with Forcing Terms

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Abstract

In this article, we solve exactly the Burgers equation with non-homogeneous terms using the Cole–Hopf type transformation. The Cole–Hopf transformation changes the classical Burgers equation to a linear equation. We modify the Cole–Hopf transformation, introducing a time dependent function to linearise the non-homogeneous Burgers equations. We consider a group of non-homogeneous terms based on the function introduced in the modified Cole–Hopf transformation. We present a list of twenty five exact solutions for non-homogeneous Burgers equations. Our study reveals that the use of the Cole–Hopf type transformation combined with the exact solutions of the heat equation can provide the exact solutions for non-homogeneous Burgers equations. This indicates the potential of the Cole–Hopf type transformation in solving nonlinear partial differential equations.

Keywords: Non-homogeneous Burgers equation, Generalized Cole–Hopf transformation, Exact solutions

1 Introduction

The nonhomogeneous Burgers equation

$$Y_t + YY_x = Y_{xx} + G(x,t), \tag{1}$$

is an extended version of the classical Burgers equation

$$Y_t + YY_x = Y_{xx}.$$

This equation is an outcome of balance of nonlinear convection, viscous diffusion, and external forcing term. It appears in different physical settings, including fluid turbulence, traffic flow, and wave motion in plasmas (see [1–4]). The non-homogeneous Burgers-type equations arise in the study of electrohydrodynamical models. Moreau and Vallée [5] derived the forced Burgers equation

$$\partial_t Y - Y\partial_{\bar{x}} Y = \bar{\nu}\partial_{\bar{x}\bar{x}} Y - \bar{x}\bar{S} + c(t),$$

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which describes the evolution of a one-dimensional electric field subject to elastic forcing. In a related setting, they analyzed (1) with a linear force $G(x,t) = -k^2 x$, interpreting it as a model for electron motion under viscosity and an imposed electric field. This formulation has since become a reference point for understanding how forcing interacts with the intrinsic nonlinearity of the Burgers-type equations.

Exact and analytical solutions of forced Burgers equations have been constructed in a number of special cases. Eule and Friedrich [6] showed that when the forcing takes the form $G(x,t) = g(t)x$, the equation can be reduced to the homogeneous Burgers equation, allowing explicit solutions to be written down. Such reductions are closely connected with models of stretched vortices in hydrodynamic flows. A more general nonhomogeneous setting was examined by Zhang *et al.* [7] in connection with extensions of the Ornstein–Uhlenbeck process. Rao and Yadav [8] derived explicit solutions of (1) with the time-dependent linear forcing $G(x,t) = \frac{kx}{(2\beta t+1)^2}$, under certain unbounded initial conditions, and investigated the corresponding large-time asymptotic behavior. In a subsequent work [9], they focused on the simpler linear forcing $G(x,t) = kx$, considering bounded and compactly supported initial data and again emphasizing asymptotic regimes. Salas [10] obtained new classes of solutions for $G(x,t) = g(t)x$ through a generalized Cole–Hopf transformation, while Bai and Zhao [11] applied the extended homogeneous balance method to construct rational and solitary wave solutions.

Further generalizations have also been proposed. Zolaa *et al.* [7] studied the equation

$$\partial_t u(x,t) = D\partial_x^2 u(x,t) - \partial_x\big[F(x,t)u(x,t)\big] - ku(x,t)\partial_x u(x,t) + \beta(x,t),$$

where $F(x,t)$ represents an external drift and $\beta(x,t)$ an additional forcing term. In the absence of β, they considered linear drifts of the form $F(x,t) = k_1(t) - k_2(t)x$, obtaining solutions that extend the classical Ornstein–Uhlenbeck process. Depending on the time dependence of k_1 and k_2, the solutions either approach stationary profiles or exhibit anomalous spreading. They also treated cases with $\beta(x,t) = \beta_1(t) + \beta_2(t)x$ and $\beta(x,t) = \beta_3 x - \beta_4/x^3$, for which exact solutions were derived. Connections with other areas have been explored as well. Halberg [12] related nonhomogeneous Burgers equations to stationary Schrödinger problems, which allowed the construction of time-dependent solutions, while Xu *et al.* [13] proposed a generalized Hopf–Cole transformation reducing certain forced Burgers equations to heat equations under specific constraints.

Due to various limitations in analytical approach to solve extended Burgers equation, researchers devoted substantial effort to numerical methods. Iyanda *et al.* [14] developed a four-step numerical algorithm for both initial value problem (IVP) and initial boundary value problem (IBVP) posed for Burgers equations with source terms. Sarboland *et al.* [15] proposed two numerical schemes for (1) and discussed their accuracy by comparing with available exact solutions.

The noisy Burgers equation is a particular form of non-homogeneous Burgers equation which appears to models various aspects in stochastic process theory. Fogedby [16] studied soliton solutions of the noisy Burgers equation using the Martin–Siggia–Rose functional formalism in the context of growth morphology. Emmerich and Kahng [17] investigated noisy Burgers equation in context of discrete stochastic effects. Ablowitz and Lillo [18] studied asymptotic behavior of (1) with a source $G(x,t) = \delta'(x)F(t)$. Balogh *et al.* [19] studied stationary solutions of a IBVP posed for forced viscous Burgers' equation where forcing term is a function of space variable. Recently, Jenaliyev *et al.* [21, 22] established the existence of the solutions of BVPs posed for non-homogeneous Burgers equation in Sobolev space and Amangaliyeva *et al.* [23] investigated related integral equations in weighted L^∞ spaces.

Benton and Platzman [24] compiled thirty-five distinct solutions of the classical Burgers equation. Their work demonstrates that the equation (1) without forcing term exhibits a rich range of solution behaviors. Motivated by this observation, we extend the discussion to the non-homogeneous Burgers equation and present a collection of twenty-five exact solutions for this extended equation.

The paper is organized as follows. In Section 2, we derive exact solutions for the non-homogeneous Burgers equation. Section 3 presents concluding remarks.

2 Solution of the Nonhomogeneous Burgers Equation

We consider the one dimensional nonhomogeneous Burgers equation of the form:

$$Y_t + YY_x = Y_{xx} + G(x,t), \tag{2}$$

$G(x,t)$ represents a given non-homogeneous term. To obtain exact solutions, we use a generalized Cole–Hopf transformation

$$Y(x,t) = -2\frac{v_x}{v} + F(t). \tag{3}$$

Differentiating equation (3) with respect to t and x, we obtain

$$Y_t = 2\left(\frac{v_x v_t - v v_{xt}}{v^2}\right) + F'(t), \tag{4}$$

$$Y_x = -2\left(\frac{v_{xx}v - v_x^2}{v^2}\right), \tag{5}$$

$$Y_{xx} = -2\left(\frac{v^2 v_{xxx} - 3v v_x v_{xx} + 2v_x^3}{v^3}\right). \tag{6}$$

We substitute equations (4), (5), and (6) into the nonhomogeneous Burgers equation (2). Thus, we obtain

$$\begin{aligned} &2\left(\frac{v_x v_t - v v_{xt}}{v^2}\right) + F'(t) + 4\left(\frac{v_{xx}v_x v - v_x^3}{v^3}\right) \\ &\quad - 2F(t)\left(\frac{v_{xx}v - v_x^2}{v^2}\right) = -2\left(\frac{v^2 v_{xxx} - 3v v_x v_{xx} + 2v_x^3}{v^3}\right) + G(x,t). \end{aligned}$$

This simplifies to:

$$\left(\frac{v_x v_t - v v_{xt}}{v^2}\right) + \frac{F'(t)}{2} - F(t)\left(\frac{v_{xx}v - v_x^2}{v^2}\right) = \frac{v_x v_{xx} - v v_{xxx}}{v^2} + \frac{G(x,t)}{2}. \tag{7}$$

Integrating both sides of equation (7) with respect to x from 0 to x, we get:

$$\frac{v_t - v_{xx}}{v} = \int_0^x \left[\frac{1}{v^2}\left(\frac{F'(t)v^2}{2} - F(t)(v_{xx}v - v_x^2)\right) - \frac{G(\xi,t)}{2}\right] d\xi.$$

Let us define:

$$\mathcal{F}(x,t) = -\frac{1}{v^2}\left(\frac{F'(t)v^2}{2} - F(t)(v_{xx}v - v_x^2)\right) + \frac{G(x,t)}{2}.$$

Solving the equation $\mathcal{F}(x,t) = 0$ for $G(x,t)$, we get

$$G(x,t) = F'(t) - 2F(t)\frac{\partial}{\partial x}\left(\frac{v_x}{v}\right). \tag{8}$$

Illustrative Examples

We now present two specific examples to illustrate the utility of the transformation method.

Table 1 List of solutions-Part A

Case	$G(x,t)$	$Y(x,t)$
1	$F'(t)+\frac{2F(t)}{x^2}$	$F(t)-\frac{2}{x}$
2	$F'(t)-2F(t)\frac{4t-2x^2}{(x^2+2t)^2}$	$F(t)-\frac{4x}{x^2+2t}$
3	$F'(t)+6F(t)\frac{x^4+12t^2}{(x^3+6xt)^2}$	$F(t)-\frac{6x^2+12t}{x^3+6xt}$
4	$F'(t)-2F(t)\frac{12x^2+24t}{x^4+12x^2t+12t^2}+2F(t)\left(\frac{4x^3+24xt}{x^4+12x^2t+12t^2}\right)^2$	$F(t)-\frac{8x^3+48xt}{x^4+12x^2t+12t^2}$
5	$F'(t)-2F(t)\frac{20x^3+120xt}{x^5+20x^3t+60xt^2}+2F(t)\left(\frac{5x^4+60xt+60t^2}{x^5+20x^3t+60xt^2}\right)^2$	$F(t)-\frac{10x^4+120x^2t+120t^2}{x^5+20x^3t+60xt^2}$
6	$F'(t)-2F(t)\frac{30x^4+360x^2t+360t^2}{x^6+30x^4t+180x^2t^2+120t^3}+2F(t)\left(\frac{6x^5+120x^3t+360xt^2}{x^6+30x^4t+180x^2t^2+120t^3}\right)^2$	$F(t)-\frac{12x^5+240x^3t+720xt^2}{x^6+30x^4t+180x^2t^2+120t^3}$
7	$F'(t)-2F(t)\frac{42x^5+840tx^3+2520t^2x}{x^7+42tx^5+420t^2x^3+840t^3x}+2F(t)\left(\frac{7x^6+210tx^4+1260t^2x^2+840t^3}{x^7+42tx^5+420t^2x^3+840t^3x}\right)^2$	$F(t)-\frac{14x^6+420tx^4+2520t^2x^2+1680t^3}{x^7+42tx^5+420t^2x^3+840t^3x}$
8	$F'(t)+2\mu^2F(t)\sec^2(\mu x)$	$F(t)+2\mu\tan(\mu x)$
9	$F'(t)+2\mu^2F(t)\csc^2(\mu x)$	$F(t)-2\mu\cot(\mu x)$
10	$F'(t)+2\mu^2F(t)\sec^2(\mu x-2\mu^2t)$	$F(t)+2\mu(1+\tan(\mu x-2\mu^2t))$
11	$F'(t)+2\mu^2F(t)\csc^2(\mu x-2\mu^2t)$	$F(t)+2\mu(1-\cot(\mu x-2\mu^2t))$
12	$F'(t)+F(t)\times\frac{\sqrt{\pi}x\exp\left(-\frac{x^2}{4t}\right)\operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right)+2t^{1/2}\exp\left(-\frac{x^2}{2t}\right)}{\pi t^{3/2}\operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right)^2}$	$F(t)-\frac{2\exp\left(\frac{-x^2}{4t}\right)}{\sqrt{\pi t}\operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right)}$

Case 1: $v(x,t)=x$

Substituting into equation (3) and (8) yields:

$$Y(x,t)=-\frac{2}{x}+F(t), \tag{9}$$

$$G(x,t)=F'(t)+\frac{2F(t)}{x^2}. \tag{10}$$

Case 2: $v(x,t)=x^2+2t$

For this choice, the corresponding expressions become:

$$Y(x,t)=-\frac{4x}{x^2+2t}+F(t), \tag{11}$$

$$G(x,t)=F'(t)-4F(t)\cdot\frac{2t-x^2}{(x^2+2t)^2}. \tag{12}$$

In Table 1 and Table 2, we list several exact solutions of the nonhomogeneous Burgers equation. Different choices of the exact solutions of corresponding linear equation are considered. Each row

gives the corresponding forcing term $G(x,t)$. It also gives the resulting solution $Y(x,t)$ obtained using the generalized Cole–Hopf transformation.

Table 2 List of solutions-Part B

Case	$G(x,t)$	$Y(x,t)$
13	$F'(t)-F(t)\dfrac{\sqrt{\pi}x\exp\left(-\dfrac{x^2}{4t}\right)\operatorname{erfc}\left(\dfrac{x}{2\sqrt{t}}\right)-2\sqrt{t}\exp\left(-\dfrac{x^2}{2t}\right)}{\pi t^{3/2}\operatorname{erfc}\left(\dfrac{x}{2\sqrt{t}}\right)^2}$	$F(t)+\dfrac{2\exp\left(-\dfrac{x^2}{4t}\right)}{\sqrt{\pi t}\operatorname{erfc}\left(\dfrac{x}{2\sqrt{t}}\right)}$
14	$F'(t)-2F(t)\dfrac{\pm e^{t+x}}{(\pm1+\mathrm{e}^{t+x})^2}$	$F(t)-\dfrac{2\mathrm{e}^{t+x}}{\pm1+\mathrm{e}^{t+x}}$
15	$F'(t)-2F(t)\dfrac{\pm\mathrm{e}^{t-x}}{(\pm1+\mathrm{e}^{t-x})^2}$	$F(t)+\frac{2\mathrm{e}^{t-x}}{\pm1+\mathrm{e}^{t-x}}$
16	$F'(t)-2F(t)\dfrac{\mathrm{e}^{2t}\pm\mathrm{e}^{t}\cosh x}{(\pm1+\mathrm{e}^{t}\cosh x)^2}$	$F(t)-\dfrac{2\mathrm{e}^{t}\sinh x}{\pm1+\mathrm{e}^{t}\cosh x}$
17	$F'(t)-2F(t)\dfrac{\pm\mathrm{e}^{t}\sinh x-\mathrm{e}^{2t}}{(\pm1+\mathrm{e}^{t}\sinh x)^2}$	$F(t)-\dfrac{2\mathrm{e}^{t}\cosh x}{\pm1+\mathrm{e}^{t}\sinh x}$
18	$F'(t)+2F(t)\dfrac{\mathrm{e}^{-2t}\pm\mathrm{e}^{-t}\cos x}{(\pm1+\mathrm{e}^{-t}\cos x)^2}$	$F(t)+\dfrac{2\mathrm{e}^{-t}\sin x}{\pm1+\mathrm{e}^{-t}\cos x}$
19	$F'(t)+2F(t)\dfrac{\mathrm{e}^{-2t}\pm\mathrm{e}^{-t}\sin x}{(\pm1+\mathrm{e}^{-t}\sin x)^2}$	$F(t)-\frac{2\mathrm{e}^{-t}\cos x}{\pm1+\mathrm{e}^{-t}\sin x}$
20	$F'(t)+\frac{F(t)}{t}$	$F(t)+\dfrac{x}{t}$
21	$F'(t)+F(t)\times\dfrac{\pm\left(\dfrac{1}{t^{3/2}}-\dfrac{x^2}{2t^{5/2}}\right)\exp\left(\dfrac{x^2}{4t}\right)+\dfrac{1}{t^2}\exp\left(\dfrac{x^2}{2t}\right)}{\left(\pm1+t^{-1/2}\exp\left(-\dfrac{x^2}{4t}\right)\right)^2}$	$F(t)+\dfrac{xt^{-3/2}\exp\left(-\dfrac{x^2}{4t}\right)}{\pm1+t^{-1/2}\exp\left(-\dfrac{x^2}{4t}\right)}$
22	$F'(t)+\dfrac{F(t)}{t^2}\left(t+\dfrac{2}{\sinh^2\left(\dfrac{x}{t}\right)}\right)$	$F(t)+\dfrac{x}{t}-\dfrac{2}{t}\coth\left(\dfrac{x}{t}\right)$
23	$F'(t)+\dfrac{F(t)}{t}-\dfrac{2F(t)}{t^2\cosh^2\left(\dfrac{x}{t}\right)}$	$F(t)+\dfrac{x}{t}-\dfrac{2}{t}\tanh\left(\dfrac{x}{t}\right)$
24	$F'(t)-\dfrac{2F(t)}{\cosh^2 x}$	$F(t)-2\tanh x$
25	$F'(t)+\dfrac{2F(t)}{\sinh^2 x}$	$F(t)-2\coth x$

3 Conclusion

In this work, we derived exact solutions for the non-homogeneous Burgers equations using an extended Cole-Hopf-type transformation. We showed that a modification in the Cole-Hopf transformation enables us to solve non-homogeneous Burgers equations. In Tables 1 and 2, we listed twenty five exact solutions for non-homogeneous Burgers equations. In general, the solution of the considered non-homogeneous Burgers equation subject to initial and boundary conditions either involves undetermined integrals or is in implicit form. However, these type of solutions can be approximated asymptotically by the exact solutions presented.

Declarations

- The authors received no specific funding for this study.

- The authors declare that they have no conflicts of interest to report regarding the present study.
- No Human subject or animals are involved in the research.
- All authors have mutually consented to participate.
- All the authors have consented the Journal to publish this paper.
- Authors declare that all the data being used in the design and production cum layout of the manuscript is declared in the manuscript.

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