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Large-Time Asymptotic Behavior of the Generalized Burgers Equation on the Infinite Line

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Abstract

This study examines a generalized Burgers equation (GBE) with step-like initial data on the infinite line. For the classical Burgers equation, the Cole–Hopf transformation facilitates the derivation of exact solutions, whose asymptotic behavior is systematically analyzed. As this transformation is not applicable to general flux functions, numerical simulations based on the method of lines are employed to investigate the large-time asymptotic behavior of the GBE. The findings reveal that the asymptotic profile evolves into either a traveling or a stationary wave, depending on the initial condition. Numerical experiments validate the large-time asymptotic behavior obtained from the analytical solution of the classical Burgers equation and demonstrate the reliability of the numerical method used. In cases where exact solutions are unavailable, the large-time asymptotic behavior is investigated through numerical analysis.

Keywords: Generalized Burgers Equation, Asymptotic behavior, Traveling wave, stationary wave, Method of lines

1 Introduction

In this work, we study the initial value problem for a generalized Burgers equation of the form

$$V_t + (H(V))_x + K(t)V_x = V_{xx}, \quad -\infty < x < \infty,\ t > 0, \tag{1}$$

$$V(x,0) = v_+ \quad \text{for } x \geq 0, \qquad V(x,0) = v_- \quad \text{for } x < 0, \tag{2}$$

where $v_- > v_+$ and $v_- > 0$. The coefficient $K(t)$ is written as $K(t) = k_0 + k_e(t)$, and the flux function $H \in \mathrm{C}^2$ satisfies $H(0) = 0$ together with the convexity condition $H''(u) > 0$ on the domain of interest.

The equation

$$V_t + VV_x + K(t)V_x = V_{xx}$$

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illustrates the interaction between nonlinear transport, diffusion, and a time-dependent external influence. Each term plays a distinct role. The nonlinear convection governs wave steepening, diffusion counteracts gradient growth, and the function $K(t)$ introduces temporal variability into the transport mechanism. Equations of this type arise in several physical and engineering contexts. As discussed in [4], Burgers-type models appear in studies of turbulence, viscous acoustics, shock formation, and magnetohydrodynamics. The additional term $K(t)V_x$ allows the model to capture effects induced by time-dependent background flows, externally imposed forcing, or fluctuating environments. The qualitative behavior of solutions depends strongly on the choice of $K(t)$, ranging from diffusive spreading to persistent traveling waves or shock-like profiles. For this reason, the equation provides a useful setting for examining how nonlinear structures persist or decay in the presence of dissipation and external modulation. It also bears connections to stochastic transport models, including the Ornstein–Uhlenbeck process, and to diffusion and wave phenomena in complex media.

Equation (1) includes several classical models as special cases. When $H(V) = V^2/2$ and $K(t) = 0$, it reduces to the standard viscous Burgers equation introduced by Burgers [1] in the context of turbulence modeling. Time-dependent extensions have been considered in various settings. Babajanov et al. [2] studied a modified Burgers equation with time-dependent coefficients, motivated by physical systems whose properties evolve over time. Such models occur in fluid dynamics, plasma physics, nonlinear optics, and wave propagation, where variations in viscosity or convection significantly affect solution behavior. Their analysis, supported by graphical representations, highlights the role of time-dependent parameters in shaping nonlinear wave dynamics. Other variants, including modified Burgers equations with flux functions of the form $H(V) = V^{s+1}/(s+1)$ for $s \geq 3$, arise in models of electromagnetic wave propagation [3]. The case $H(V) = V^2/2$ with nonzero $K(t)$ has also been linked to the Ornstein–Uhlenbeck process [4].

For the classical Burgers equation, the Cole–Hopf transformation [5, 6] provides a direct route to linearization and exact solutions. This approach does not extend to general choices of $H(V)$ and $K(t)$, and in such cases the equation remains genuinely nonlinear [7, 8]. Analytical progress has nevertheless been made without relying on linearization. Kato [9] examined self-similar solutions for flux functions of the form $H(V) = (b/2)V^2 + (c/3)V^3$. Leach [10] employed matched asymptotic expansions to study modified Burgers equations and analyzed their large-time behavior in terms of traveling wave solutions.

When exact analysis is not feasible, numerical methods provide an effective alternative. The method of lines [11, 12] has been widely used for Burgers-type equations and is adopted here following the approach in [10]. We apply this method to the generalized equation for several nonlinear flux functions, including $H(V) = \exp(V^2)$ and $H(V) = \log(\exp(V) + 1)$, in the presence of external forcing. The computations are carried out using MATLAB. Related numerical studies on finite domains may be found in [13–15]. Other numerical approaches have also been developed to address variable-coefficient convection–diffusion equations. Lattice Boltzmann models with time- and space-dependent coefficients have shown improved stability and accuracy for Burgers-type problems with variable convection [16]. Compact finite difference and implicit schemes have likewise been proposed, together with stability and error analyses, for convection–diffusion systems with variable coefficients [17, 18].

The analysis begins with the derivation of the exact solution of (1)–(2) for the quadratic flux $H(V) = V^2/2$. From this solution, the large-time asymptotic behavior of $V(x,t)$ is obtained in the form

$$V(x,t) \sim m^* - k^* \tanh\left[\frac{k^*}{2}\left(x - \int_0^t K(s)\,ds - m^* t\right)\right], \tag{3}$$

where $k^* = \frac{1}{2}(v_- - v_+)$ and $m^* = \frac{1}{2}(v_- + v_+)$. The corresponding wave position and speed are given by

$$W(t) = m^* t + \int_0^t K(\omega)\,d\omega, \qquad \dot{W}(t) = m^* + K(t).$$

The wave propagates in the positive or negative x-direction depending on the sign of $\dot{W}(t)$. A stationary wave arises asymptotically when $\dot{W}(t)$ approaches zero as $t \to \infty$. Numerical experiments for more general flux functions indicate that traveling wave structures continue to dominate the long-time behavior. In these cases, the wave position and speed are described by

$$W(t) = \frac{H(v_-) - H(v_+)}{v_- - v_+}\, t + \int_0^t K(\omega)\, d\omega, \qquad \dot{W}(t) = K(t) + \frac{H(v_-) - H(v_+)}{v_- - v_+}. \tag{4}$$

The paper is organized as follows. Section 2 presents the exact solution of the Burgers equation with an external forcing term. Section 3 examines traveling wave solutions for the generalized system (1)–(2). Numerical results are reported in Section 4, and concluding remarks are given in Section 5.

2 Solution to the IVP Equation (1)–Equation (2) via the Cole–Hopf Transformation for $H(V) = V^2/2$

In this section, the exact and asymptotic solutions of the initial value problem (1)-(2) are derived using the Cole–Hopf transformation. Introducing the transformation

$$V = -2\,\frac{Z_x}{Z},$$

and performing the coordinate shift

$$x \mapsto x - \int_0^t K(\omega)\, d\omega,$$

the governing system (1)-(2) is transformed into the following initial value problem:

$$Z_t = Z_{xx}, \tag{5}$$

$$Z(x,0) = \exp\left[-\tfrac{1}{2}\int_0^x V_i(s)\, ds\right], \tag{6}$$

where $V_i(x) = V(x,0)$ denotes the initial condition.

The solution to (5)–(6) can be expressed explicitly as

$$Z(x,t) = \int_{-\infty}^{\infty} \frac{Z_i(\sigma)}{2\sqrt{\pi t}}\, \exp\left[-\frac{x^2 - 2x\sigma + \sigma^2}{4t}\right] d\sigma, \tag{7}$$

where $Z_i(x) = Z(x,0)$. By performing differentiation of (7) with respect to x, we arrive at

$$Z_x(x,t) = -\int_{-\infty}^{\infty} \frac{(x-\sigma)}{4\sqrt{\pi}t^{3/2}}\, Z_i(\sigma)\, \exp\left[-\frac{x^2 - 2x\sigma + \sigma^2}{4t}\right] d\sigma. \tag{8}$$

Hence, the corresponding solution of (1)–(2) takes the form

$$V(x,t) = \frac{\displaystyle\int_{-\infty}^{\infty} \frac{E(x,t,\sigma)}{t}\, \exp\left[-\frac{E(x,t,\sigma)^2}{4t}\right] Z_i(\sigma) d\sigma}{\displaystyle\int_{-\infty}^{\infty} \exp\left[-\frac{E(x,t,\sigma)^2}{4t}\right] Z_i(\sigma) d\sigma}, \tag{9}$$

where $E(x,t,\sigma) = x - \int_0^t K(\omega)\, d\omega - \sigma$. In view of the initial data (2), we obtain

$$Z_i(x) = \exp\left(\tfrac{v_- x}{2}\right), \qquad x < 0, \tag{10}$$

$$Z_i(x) = \exp\!\left(-\tfrac{v_+ x}{2}\right), \qquad x \geq 0. \tag{11}$$

Substituting these into (9), the solution simplifies to

$$V(x,t) = \frac{v_- \exp\!\left(\frac{v_-^2 t}{4} - \frac{v_-\left(x - \int_0^t K(\omega)\,d\omega\right)}{2}\right) E_1(x,t) + v_+ \exp\!\left(\frac{v_+^2 t}{4} - \frac{v_+\left(x - \int_0^t K(\omega)\,d\omega\right)}{2}\right) E_2(x,t)}{\exp\!\left(\frac{v_-^2 t}{4} - \frac{v_-\left(x - \int_0^t K(\omega)\,d\omega\right)}{2}\right) E_1(x,t) + \exp\!\left(\frac{v_+^2 t}{4} - \frac{v_+\left(x - \int_0^t K(\omega)\,d\omega\right)}{2}\right) E_2(x,t)}, \tag{12}$$

where

$$E_1(x,t) = \operatorname{erfc}\left(\frac{x - \int_0^t K(\omega)\,d\omega - v_- t}{2\sqrt{t}}\right), \quad E_2(x,t) = \operatorname{erfc}\left(\frac{\int_0^t K(\omega)\,d\omega + v_+ t - x}{2\sqrt{t}}\right).$$

It can be observed that $E_1(x,t) \sim 2$ and $E_2(x,t) \sim 2$ for $t \to \infty$ when $x < \int_0^t K(\omega)\,d\omega + v_- t$ and $x > \int_0^t K(\omega)\,d\omega + v_+ t$, respectively. Hence, within the intermediate region

$$\int_0^t K(\omega)\,d\omega + v_+ t < x < \int_0^t K(\omega)\,d\omega + v_- t, \quad t \to \infty,$$

the expression in (12) reduces asymptotically to $V(x,t) \sim V_T(x,t)$, where

$$V_T(x,t) = \frac{v_- \exp\!\left(\frac{v_-^2 t}{4} - \frac{v_-}{2}\left[x - \int_0^t K(\omega)\,d\omega\right]\right) + v_+ \exp\!\left(\frac{v_+^2 t}{4} - \frac{v_+}{2}\left[x - \int_0^t K(\omega)\,d\omega\right]\right)}{\exp\!\left(\frac{v_-^2 t}{4} - \frac{v_-}{2}\left[x - \int_0^t K(\omega)\,d\omega\right]\right) + \exp\!\left(\frac{v_+^2 t}{4} - \frac{v_+}{2}\left[x - \int_0^t K(\omega)\,d\omega\right]\right)}. \tag{13}$$

The above expression can be simplified to obtain a more compact asymptotic form,

$$V_T(x,t) = m^* - k^* \tanh\left[\frac{k^*}{2}\left(x - \int_0^t K(\omega)\,d\omega - m^* t\right)\right], \tag{14}$$

where $k^* = \frac{1}{2}(v_- - v_+)$ and $m^* = \frac{1}{2}(v_- + v_+)$. The function $V_T(x,t)$ represents a travelling wave whose position and instantaneous velocity are given by

$$W(t) = \frac{v_- + v_+}{2}\,t + \int_0^t K(\omega)\,d\omega, \qquad \dot{W}(t) = \frac{v_- + v_+}{2} + K(t). \tag{15}$$

It is important to note that the travelling wave solution $V_T(x,t)$ can be represented as a function of a single variable $\tau = x - \int_0^t K(\omega)\,d\omega - m^* t$. This representation satisfies the ordinary differential equation

$$V_T''(\tau) + m^* V_T'(\tau) - V_T(\tau) V_T'(\tau) = 0.$$

This observation serves as a motivation to obtain the travelling wave solution of the generalized Burgers equation (1) for a more general expression of $H(V)$.

3 Travelling wave solution of the GBE

In this section, following the approach of [19], we analyze the travelling wave solutions of equations (1)–(2). We assume a wave profile of the form

$$V(x,t) = V_T(\tau), \qquad \tau = x - m^* t - \int_0^t K(\omega)\,d\omega - d_2.$$

Substituting this form into (1)–(2) reduces the system to the following ordinary differential equation:

$$V_T'' + m^* V_T' - H'(V_T) V_T' = 0. \tag{16}$$

Considering the initial condition (2), the travelling wave satisfies

$$V_T(\tau) \to v_\pm \quad \text{as } \tau \to \pm\infty. \tag{17}$$

By integrating equation (16) twice, we obtain

$$\int_{V_T(0)}^{V_T} \frac{d\psi}{H(\psi) - m^*\psi + d_2} = \tau. \tag{18}$$

The integral on the left-hand side is a monotonic function of $V_T(\tau)$, since $H''(V_T) > 0$ implies that the term $H(\psi) - m^*\psi + d_2$ is strictly monotonic and nonzero. Hence, the function is invertible, confirming the existence of a unique travelling wave solution $V_T(\tau)$.

Applying the boundary condition (17), the constants m^* and d_2 are determined accordingly. Hence, the travelling wave solution takes the form

$$V(x,t) = V_T\left(x - m^* t - \int_0^t K(\omega)\, d\omega - d_2\right).$$

Thus, the position and velocity of the travelling wave are given respectively by

$$W(t) = \frac{H(v_-) - H(v_+)}{v_- - v_+}\, t + \int_0^t K(\omega)\, d\omega + \frac{v_+ H(v_-) - v_- H(v_+)}{v_- - v_+}, \tag{19}$$

$$\dot{W}(t) = \frac{H(v_-) - H(v_+)}{v_- - v_+} + K(t). \tag{20}$$

The expressions above describe the propagation and temporal evolution of the travelling wave associated with the generalized Burgers equation.

In general, the integral appearing in (18) cannot be evaluated analytically for an arbitrary form of H. However, for certain specific choices of H, v_-, and v_+, an exact closed-form expression can be obtained. For example, when $H(V) = \frac{V^2}{2}$, the solution of (18) can be written explicitly as presented in (14).

For the generalized Burgers equation (1) with $H'(V) = V^s$ $(s > 1)$ and $v_+ = 0$, equation (18) reduces to

$$V_T(x,t) = v_- \left[\frac{1}{2} - \frac{1}{2}\tanh\left(\frac{sm^*}{2}\left(x - \int_0^t K(\omega)\, d\omega - m^* t - c_1\right)\right)\right]^{\frac{1}{s}}, \tag{21}$$

where $m^* = \frac{v_-^s}{s+1}$. This analytical form is expected to describe the asymptotic behaviour of the solution $V(x,t)$ as $t \to \infty$. Our numerical simulations confirm this observation, showing that the long-time numerical profile closely matches the analytical expression given by (21).

4 Numerical Investigation of the GBE

4.1 Method of Lines

To numerically investigate the generalized Burgers equation, we consider the computational domain

$$-X \le x \le X, \quad 0 \le t \le T.$$

This region is divided into a set of rectangular subdomains using the grid lines

$$x_i = il, \quad (i = 1, 2, \ldots, p+1), \qquad t_j = jn, \quad (j = 1, 2, \ldots, q+1),$$

where $l = 2X/p$ and $n = T/q$ denote the spatial and temporal step sizes, respectively.

The spatial derivative V_x at $x = x_i$ is approximated using a fourth-order finite difference formula (see [11, 12]). Similarly, higher-order spatial derivatives such as V_{xx} are computed using compact difference schemes suitable for Dirichlet boundary conditions. Applying these discretizations to the governing equation reduces it to a system of ordinary differential equations (ODEs) of the form

$$\frac{dY}{dt} = N(Y, t), \tag{22}$$

where $Y = (V_1, V_2, \ldots, V_{p+1})$ and $N(y, t) = (N_1(Y, t), N_2(Y, t), \ldots, N_{q+1}(Y, t))$. Each function $G_i(Y, t)$ represents the finite difference approximation of the spatial derivatives and the nonlinear convective term.

The resulting ODE system is integrated in time using the classical fourth-order Runge–Kutta (RK4) scheme. A detailed stability analysis of the method of lines for the Burgers equation can be found in [20], where it has been shown that the scheme is unconditionally convergent.

Now, we demonstrate the accuracy and efficiency of the numerical method (the method of lines) through a series of test problems involving different forms of the generalized Burgers equation (GBE). The study includes examples for which exact solutions are known, as well as cases where analytical solutions are not available, allowing us to assess both the validity and robustness of the numerical approach.

Problem (i): Consider the Burgers equation with time dependent coefficient

$$V_t + VV_x + \frac{V_x}{1+t} = V_{xx}, \tag{23}$$

with boundary conditions $v_- = 1.0$ and $v_+ = 0.0$. Using (12), the corresponding exact solution is expressed as

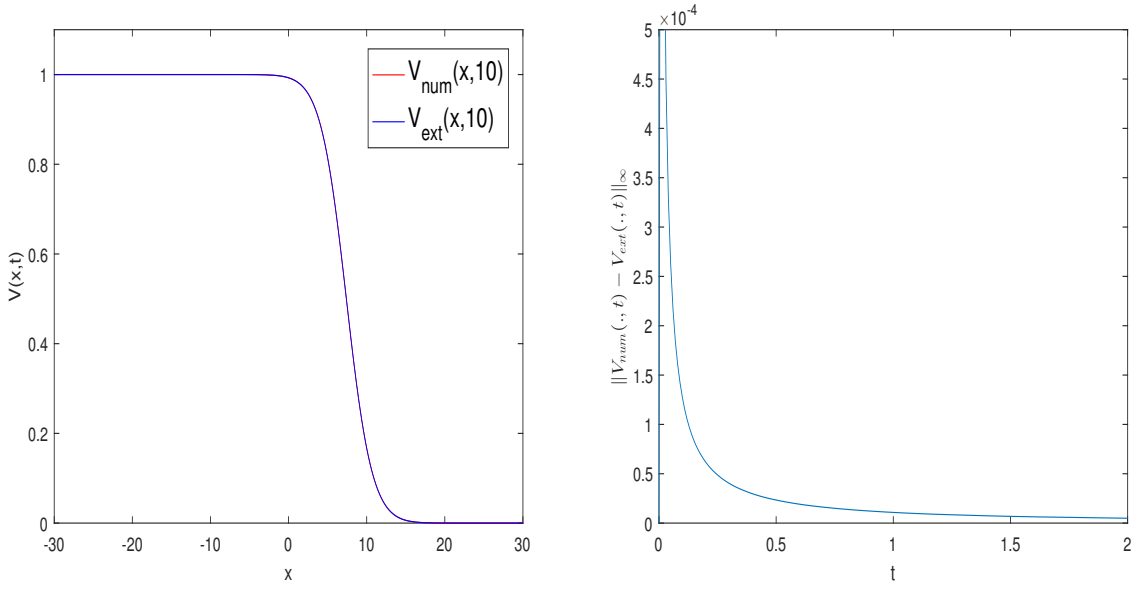
$$V(x,t) = \frac{\exp\left(\frac{t}{4} - \frac{x-\log(1+t)}{2}\right) \operatorname{erfc}\left(\frac{x-\log(1+t)-t}{2\sqrt{t}}\right)}{\exp\left(\frac{t}{4} - \frac{x-\log(1+t)}{2}\right) \operatorname{erfc}\left(\frac{x-\log(1+t)-t}{2\sqrt{t}}\right) + \operatorname{erfc}\left(\frac{\log(1+t)-x}{2\sqrt{t}}\right)}, \tag{24}$$

and its asymptotic form, as $t \to \infty$, is given by

$$V_T(x,t) = \frac{\exp\left(\frac{t}{4} - \frac{x-\log(1+t)}{2}\right)}{1 + \exp\left(\frac{t}{4} - \frac{x-\log(1+t)}{2}\right)}. \tag{25}$$

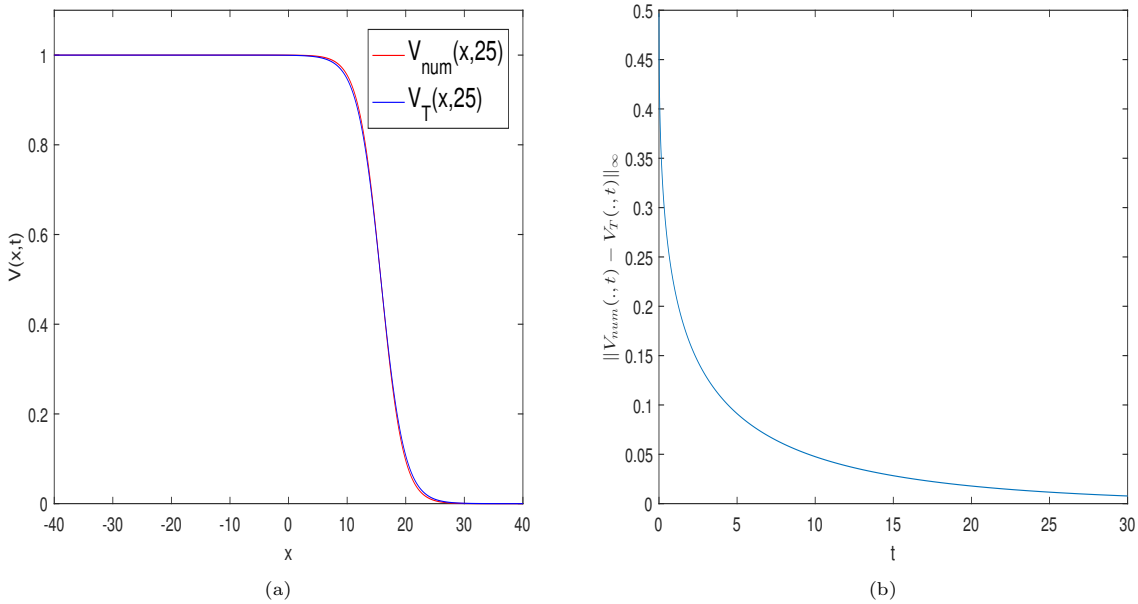
Figure 1 shows the numerical and exact solutions of (23) for $v_- = 1.0$ and $v_+ = 0.0$, illustrating their close agreement. Figure 2 compares the numerical solution with the asymptotic profile, showing convergence as time increases. Figure 3 presents a comparison between the numerical and exact wave speeds, where the maximum deviation between the exact wave speed $\dot{W}(t)$ and the computed one is 2.344×10^{-5} at $t = 30$.

Table 1 reports the maximum absolute errors $\|V_{\text{num}}(.,t) - V_{\text{ext}}(.,t)\|_\infty$ and $\|V_{\text{num}}(.,t) - V_T(.,t)\|_\infty$, which confirm the accuracy of the numerical method.



(a) (b)

Fig. 1 Comparison of exact and numerical solutions of the Burgers equation with time dependent coefficient (23) for $v_- = 1.0$ and $v_+ = 0.0$.



(a) (b)

Fig. 2 Comparison of numerical and asymptotic solutions of the GBE (23) for $v_- = 1.0$ and $v_+ = 0.0$.

Problem (ii): Next, we consider the modified Burgers equation with time dependent coefficient

$$V_t + V^2 V_x + \left(\alpha - \frac{1}{(1+t)^2}\right) V_x = V_{xx}, \tag{26}$$

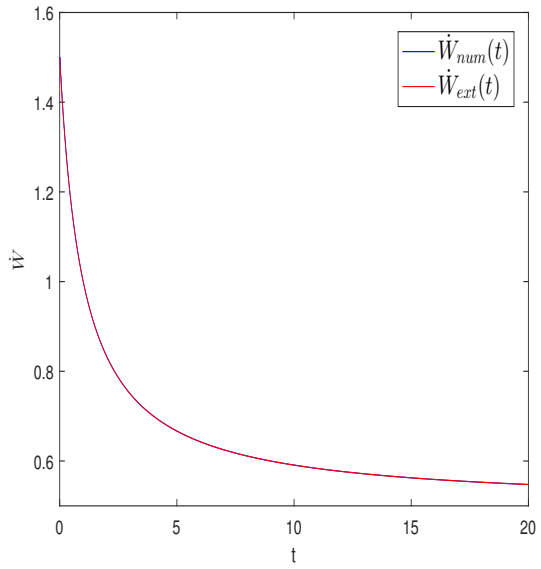


Fig. 3 Exact and numerical wave speeds for the GBE (23) with $v_- = 1.0$ and $v_+ = 0.0$.

Table 1 Comparison of the maximum errors between the numerical solution V_{num}, the exact solution (24), and the asymptotic solution (25) for $v_- = 1.0$ and $v_+ = 0.0$.

t	$\|V_{\mathbf{num}}(.,t) - V_{\mathbf{ext}}(.,t)\|_\infty$	$\|V_{\mathbf{num}}(.,t) - V_T(.,t)\|_\infty$
1	1.08091×10^{-5}	0.21784
5	1.68276×10^{-6}	0.09117
10	5.90438×10^{-7}	0.04761
15	6.38033×10^{-7}	0.02830
20	8.08586×10^{-7}	0.01784
25	8.87580×10^{-7}	0.01164
30	8.16327×10^{-7}	0.00776

subject to $v_- = 1.0$ and $v_+ = 0.0$. The large-time asymptotic solution can be written as

$$V_T(x,t) = \left[\frac{1}{2} - \frac{1}{2}\tanh\left(\frac{1}{3}\big(x - (\alpha + 1/3)t - \frac{1}{1+t} - r\big)\right)\right]^{\frac{1}{2}}. \tag{27}$$

Figure 4(a)–(c) displays the convergence of the numerical solution of (26) toward the stationary profile (27) with $r = -3.07$ as $t \to \infty$ for $\alpha = -1/3$.

Problem (iii): Finally, consider the generalised Burgers equation

$$V_t + \left(\mathrm{e}^{V^2}\right)_x + 0.2V_x = V_{xx}, \tag{28}$$

with initial conditions $v_- = 0.5$ and $v_+ = 0.0$.

Figure 5 illustrates the evolution of the travelling wave propagating in the positive x-direction. The computed wave speed converges to 0.7681, which agrees with the theoretical value predicted by (20).

Next, we examine the generalized Burgers equation of the form

$$V_t + \left(\log(\mathrm{e}^V + 1)\right)_x - V_x = V_{xx}, \tag{29}$$

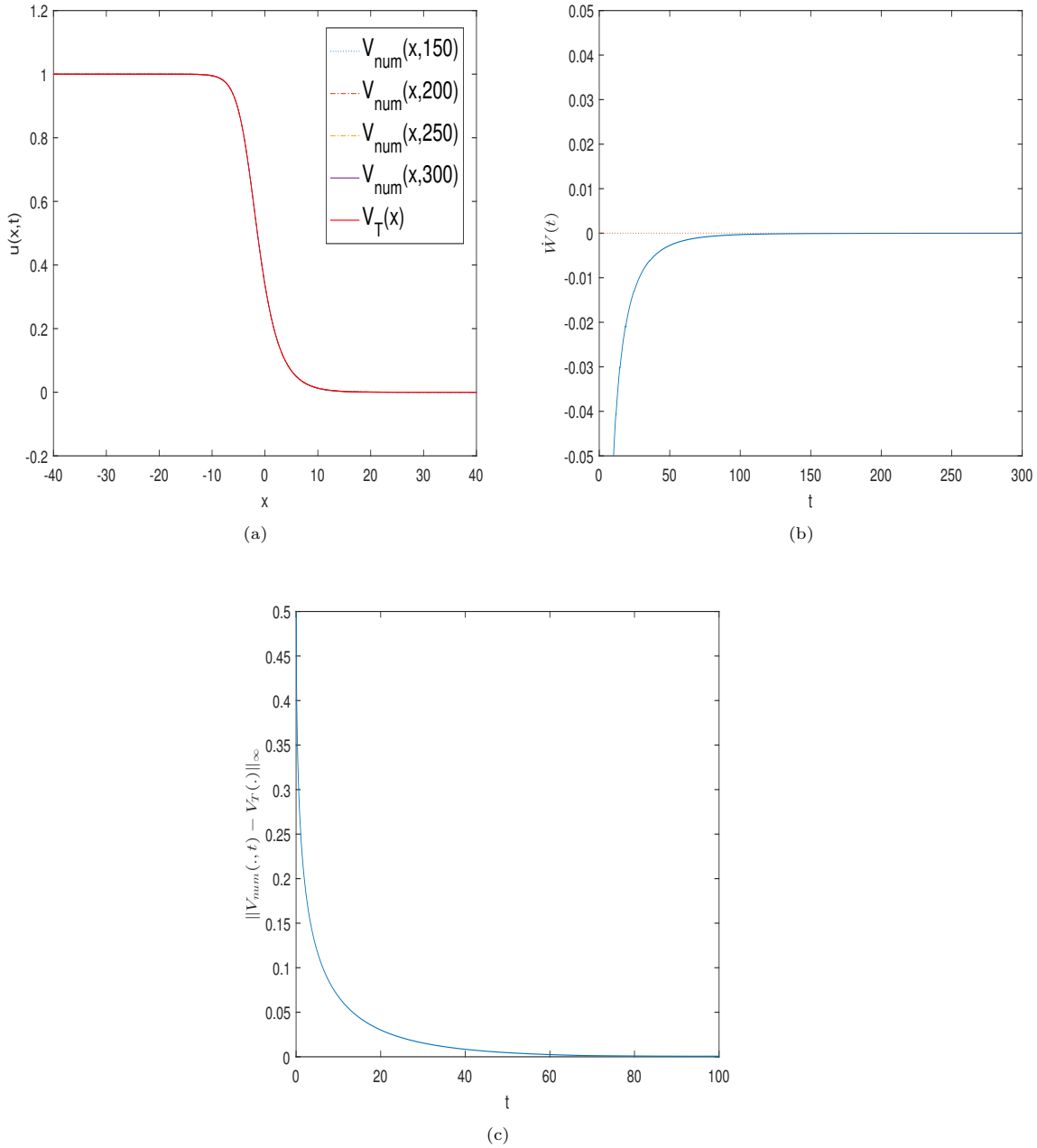
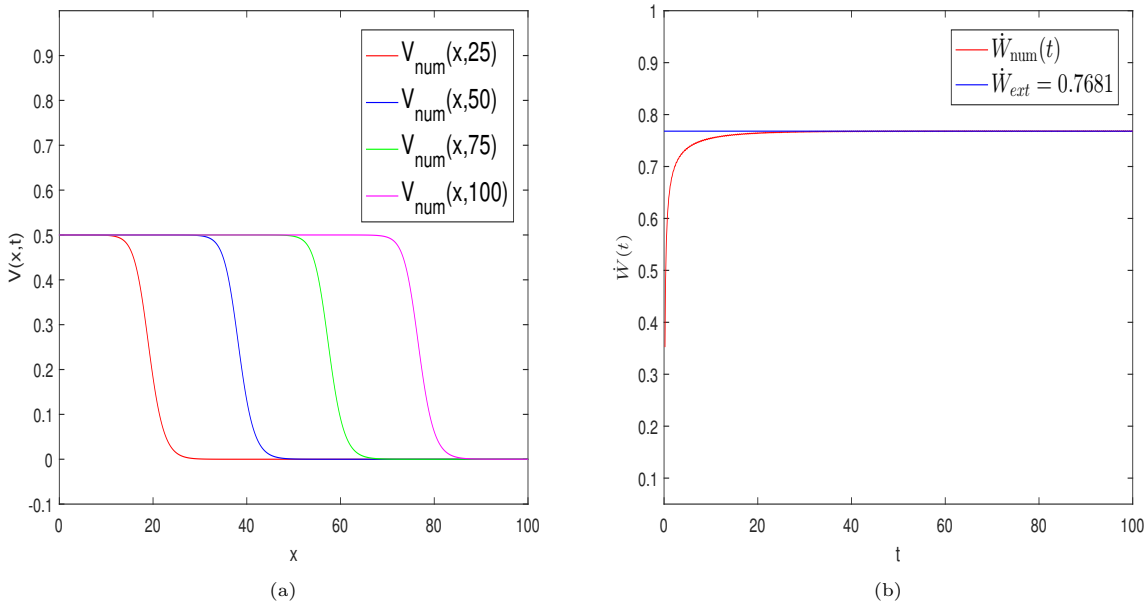


Fig. 4 (a) Numerical solutions of (26) at $t = 150,\ 200,\ 250,\ 300$ along with the stationary solution (27) at $t = 300$. (b) Tendency of the wave speed to reach zero. (c) Maximum error between the numerical and asymptotic solutions of (26).

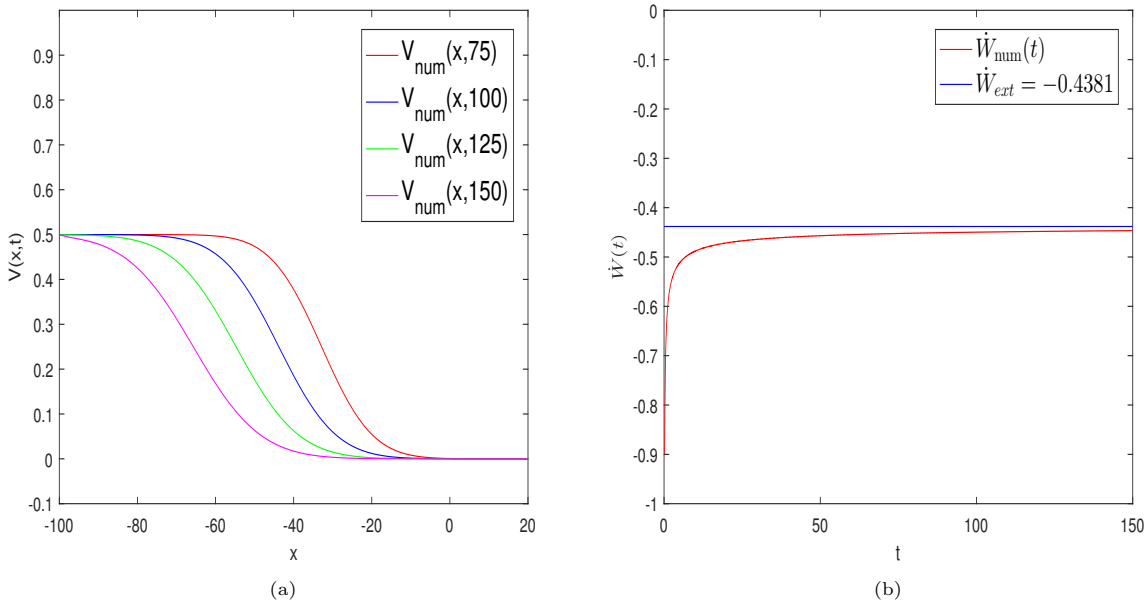
with boundary conditions $v_- = 0.5$ and $v_+ = 0.0$.

Figure 6 depicts the evolution of the travelling wave associated with (29). The wave propagates along the negative x-direction with a negative velocity. The computed wave speed converges to -0.4381, which is in excellent agreement with the theoretical prediction obtained from (20).



(a) (b)

Fig. 5 (a) Numerical solutions of (28) with $v_- = 0.5$ and $v_+ = 0.0$ at $t = 25,\ 50,\ 75,\ 100$. (b) Tendency of the wave speed to reach 0.7681.



(a) (b)

Fig. 6 (a) Numerical solutions of (29) with $v_- = 0.5$ and $v_+ = 0.0$ at $t = 75,\ 100,\ 125,\ 150$. (b) Tendency of the wave speed to reach -0.4381, showing agreement with the theoretical value.

5 Conclusions

In this study, we examined the generalized Burgers equation with a time-dependent variable coefficient Equation (1) using both analytical and numerical approaches. An exact solution was derived

for the equation, from which the corresponding wave position and wave speed were determined. In particular, an explicit travelling wave solution Equation (21) was obtained for the modified Burgers equation when $v_+ = 0$.

The method of lines was employed to compute the numerical solution, and all computations were carried out in MATLAB. The comparison between the analytical and numerical results, as shown in Table 1, demonstrates excellent agreement, with only a small numerical error—confirming the reliability and accuracy of the adopted numerical scheme. Furthermore, the numerical experiments illustrate that the solution of the generalized Burgers Equation (1) converges to a travelling wave form. The computed wave speeds for different choices of $H(V) = V^3/3$, $\exp(V^2)$, and $\log(\exp(V)+1)$ are consistent with the theoretical predictions given in Equation (20).

Declarations

- The authors received no specific funding for this study.
- The authors declare that they have no conflicts of interest to report regarding the present study.
- No Human subject or animals are involved in the research.
- All authors have mutually consented to participate.
- All the authors have consented the Journal to publish this paper.
- Authors declare that all the data being used in the design and production cum layout of the manuscript is declared in the manuscript.

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