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Function Projective Synchronization of Identical and Non-identical Chaotic System through Tracking Control Scheme

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Abstract

In this article, we discussed the function projective synchronization of similar Li chaotic systems and non-similar Lu and T chaotic systems utilising tracking control schemes. Numerical simulations have performed with the help of the Adams-Bashforth-Moulton method, and its solutions are represented graphically.

Keywords: Function projective synchronizing (FPS), Li, Lu and T chaotic systems, feedback control scheme, Adams-Bashforth-Moulton method.

1 Introduction

Chaos synchronization, or perhaps we should say chaotic synchronization, is a procedure when more than two chaotic systems, whether similar or non-similar, adapt a certain motional feature to a distributed behaviour as a outcome of pairing or force. Since chaotic systems are delicate to effects in initial situations and system characteristics., it may appear difficult to synchronize them. The synchronization of complex chaotic systems has significant applications in secure communication as a result of this feature. Numerous researchers have long been interested in and carefully researched the synchronization scenario. Pecora and Carroll [1] conducted the first analysis of chaos synchronisation in 1990. Pecora and Carroll [2] also introduced the idea of conditional Lyapunov exponents and provided the concept of controlling chaotic driving. The study of the synchronisation of chaotic systems dates back to the early 1980s, but in the past two decades it has gained popularity because to its potential applications in biological systems, nano-oscillators, secure communications and other fields [3–8]. Projective synchronization synchronizes the drive and response

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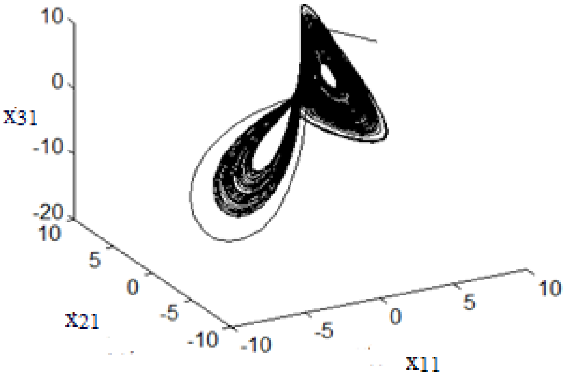

Fig. 1: Chaotic Attractor of Li System (1)

systems equal to a constant scaling factor. This proportionality feature of projective synchronization can be used to extend binary digital communication to M-nary digital communication for achieving fast communication [9, 10]. This is one of many types of chaos synchronization that has received a lot of attention recently [11–15]. FPS is extended form of projective synchronization.

The FPS system's appeal arises from its ability to synchronize the drive and response systems equal to a scaling function rather than a fixed.Therefore, its utilization to give higher error dynamical system unpredictability, which can improve communication security. In contrast to projective synchronization, this proportionate property can be employed to change binary digital communication to M-nary digital communication. Recently, many authors have studied the Function projective synchronization [16–18] of identical and non identical chaotic systems. Motivated by the preceding considerations, the goal of this article is to investigate the FPS of identical and non-identical chaotic system through tracking control scheme. But to the best of author's knowledge the FPS of chaotic systems using tracking control method have yet been studied by some researcher. Here, numerical simulations are performed using the Adams-Bashforth-Moulton method [19, 20] and graphically displayed to show the effectiveness of the suggested strategy for several specific scenarios. This article is structured as follows. Our discussion of System description is covered in Section 2. Section 3 presents the problem formulation for the function projective synchronization of a chaotic system using a tracking control strategy. In sections 4 and 5, function projective synchronization of similar and dissimilar chaotic systems is covered. The effectiveness of the suggested methodology has proved through numerical simulations in sections 4.1 and 5.1, respectively. In Section 6, the overall conclusion of study is presented.

2 System descriptions

Li system [21] is derived from the Lorenz system and is defined as

$$\begin{aligned} D_t x_{11} &= a_{11}(x_{21} - x_{11}) \\ D_t x_{21} &= -x_{21} + x_{11}x_{31} \\ D_t x_{31} &= b_{11} - x_{11}x_{21} - c_{11}x_{31} \end{aligned} \tag{1}$$

where the state variables are x_{11}, x_{21}, x_{31} and the real parametric values are $a_{11} = 5, b_{11} = 16, c_{11} = 1$. The chaotic attractor in 3-dimensional space is depicted through Figure 1 with initial condition $(5, 6, -4)$.

The Lu chaotic system was proposed by Lu et al., 2002 [22] and given as

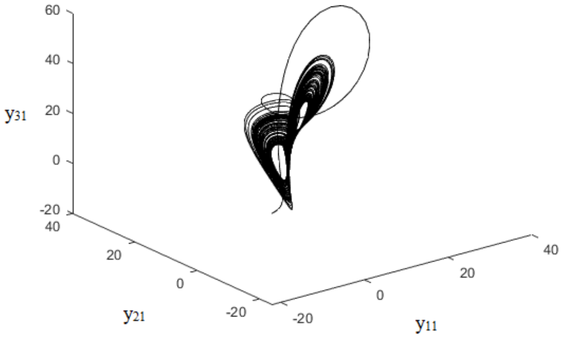

Fig. 2: Phase portraits of system (2) in space

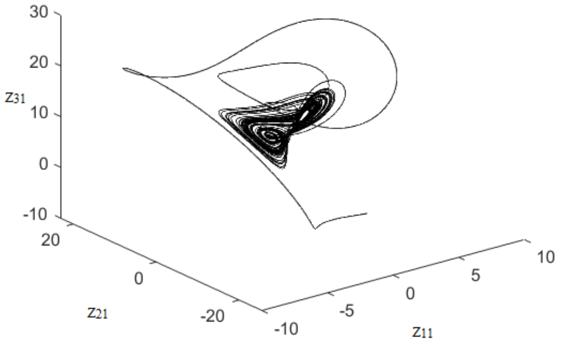

Fig. 3: Phase portraits of T- system (3) in space

$$\begin{aligned} D_t y_{11} &= a_{21}(y_{21} - y_{11}) \\ D_t y_{21} &= c_{21} y_{21} - y_{11} y_{31} \\ D_t y_{31} &= y_{11} y_{21} - b_{21} y_{31} \end{aligned} \tag{2}$$

where y_{11}, y_{21}, y_{31} are state variables, and parameters of the system a_{21}, b_{21}, c_{21} are real. The system (2) represents chaotic behavior for this constant $a_{21} = 36, b_{21} = 33, c_{21} = 20$ with initial condition (-2,3,-8).

Jiang et al., 2010 [23] presents a brand-new three-dimensional autonomous continuous chaotic dynamical T-system, which is defined as

$$\begin{aligned} D_t z_{11} &= a_{31}(z_{21} - z_{11}) \\ D_t z_{21} &= (c_{31} - a_{31}) z_{11} - a_{31} z_{11} z_{31} \\ D_t z_{31} &= -b_{31} z_{31} + z_{11} z_{21} \end{aligned} \tag{3}$$

where z_{11}, z_{21}, z_{31} are state variables, and a_{31}, b_{31}, c_{31} are parameters of the system. When parameters are taken as $(a_{31}, b_{31}, c_{31}) = (2.6, 0.4, 28)$, the maximal Lyapunov exponent of system is 0.37, which shows that the system exhibits chaos.

3 Function projective synchronization (FPS) of chaotic system through tracking control scheme

The drive and response of chaotic systems can be written as

$$D_t\varepsilon = F(\varepsilon) \tag{4}$$

$$D_t\eta = G(\eta) + \psi(\eta, \varepsilon) \tag{5}$$

where

$$\varepsilon \in \mathbb{R}^m \quad \text{and} \quad \eta \in \mathbb{R}^m$$

are m-dimensional state vectors of the equation (4) and equation (5) respectively.

$F, G : \mathbb{R}^m \to \mathbb{R}^m$ are two continuous nonlinear functions, $\psi(\eta, \varepsilon) : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ is a controller to be determined later. If there exists a controller $\psi(\eta, \varepsilon)$ such that

$$\lim_{t\to\infty} \|\zeta\| = \lim_{t\to\infty} \|\eta - \kappa(\varepsilon)\varepsilon\| = 0, \tag{6}$$

where $\zeta = (\zeta_1, \zeta_2, \dots, \zeta_m)^T \in \mathbb{R}^m$ is a state vector, $\kappa(\varepsilon) = \operatorname{diag}(k_1(\varepsilon), k_2(\varepsilon), \dots, k_m(\varepsilon))$, $(j = 1, 2, \dots, m)$, are called continuous scaling functions and such synchronization is called FPS. Using the concept of tracking control, in order to accomplish the purpose of equation (6), it is assumed that the function

$$\psi(\eta, \varepsilon) = D_t(\kappa(\varepsilon)\varepsilon) - G(\kappa(\varepsilon)\varepsilon) + \mathcal{B}(\varepsilon, \eta)\zeta, \tag{7}$$

is an appropriate controller. Substituting the equation (7) into the equation (5), the error dynamics are as follows

$$D_t\zeta = -(\mathcal{A}(\varepsilon, \eta) + \mathcal{B}(\varepsilon, \eta))\zeta, \tag{8}$$

where $\mathcal{B}(\varepsilon, \eta) = G(\eta) - G(\kappa(\varepsilon)\varepsilon)$, $\mathcal{B}(\varepsilon, \eta) \in \mathbb{R}^{m\times m}$ is also a polynomial matrix.

Then, the FPS between the systems (4) and (5) is transformed into the analysis of the asymptotical stability of zero solution of the error system (6).

For the equation (4) and (5) there exists a control vector $\mathcal{B}(x, y)\zeta$ such that FPS between (4) and (5) can achieved if

$$(\mathcal{A} + \mathcal{B}) + (\mathcal{A} + \mathcal{B})^T = -Q_1, \tag{9}$$

where $P_1, Q_1 \in \mathbb{R}^{m\times m}$ are real symmetric positive definite matrices and T stands for conjugate transpose of a matrix [24].

4 Function projective synchronization between identical chaotic systems

Consider Li-system as the drive system as

$$D_t x_{11} = c_{11}(x_{21} - x_{11}) \tag{6}$$
$$D_t x_{21} = x_{11} + x_{11}x_{31} \tag{7}$$
$$D_t x_{31} = b_{11} - x_{11}x_{21} - c_{11}x_{31}, \tag{10}$$

and Li-system as response system as

$$D_1 y_{11} = a_{11}(y_{21} - y_{11}) + \mu_1 \tag{8}$$
$$D_1 y_{21} = -y_{21} + y_{11}y_{31} + \mu_2 \tag{9}$$
$$D_1 y_{31} = b_{11} - y_{11}y_{21} - c_{11}y_{31} + \mu_3, \tag{10}$$

where, μ_1, μ_2, μ_3 are control functions.

Then, the error states become

$$e_1 = y_{11} - k_1(X)x_{11} \tag{11}$$
$$e_2 = y_{21} - k_2(X)x_{21} \tag{12}$$
$$e_3 = y_{31} - k_3(X)x_{31}. \tag{13}$$

With the suitable controller, we can get the error system as

$$D_t e = (M_1(X,Y) + M_2(X,Y))e,$$

where

$$M_1(X,Y) = \begin{pmatrix} -5 & 5 & 0 \\ y_{31} & -1 & \hat{k}_1(x)x_{11} \\ -y_{21} & -\hat{k}_1(x)x_{11} & -1 \end{pmatrix}$$

and

$$M_2(X,Y) = \begin{pmatrix} 0 & -5 - y_{31} & y_{21} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence, we can obtain

$$M_1(X,Y) + M_2(X,Y) = \begin{pmatrix} -5 & -y_{31} & y_{21} \\ y_{31} & -1 & \hat{k}_1(x)x_{11} \\ -y_{21} & -\hat{k}_1(x)x_{11} & -1 \end{pmatrix}.$$

Choosing positive definite real symmetric matrix $P_1 = \mathrm{diag}(1,2,1)$, we get

$$[M_1(X,Y) + M_2(X,Y)]^T P_1 + P_1[M_1(X,Y) + M_2(X,Y)] = \mathrm{diag}(-10,-4,-2).$$

Choosing real symmetric positive definite matrix $Q_1 = \mathrm{diag}(10,4,2)$, we get

$$[M_1(X,Y) + M_2(X,Y)]P_1 + P_1[M_1(X,Y) + M_2(X,Y)]^T = -Q_1.$$

Thus, the chaotic Li-system achieves the FPS.

4.1 Numerical simulation

In numerical simulations, choose the scaling function $\kappa(X) = \mathrm{diag}(x_{11}x_{31}, 1 + x_{31}, x_{21} - 3)$, and the error state

$$E = \sqrt{\sum_{j=1}^{3}(y_j - \hat{k}_j(X)x_j)^2}$$

$(2, 0.9, -0.6)$ and $(0.6, -0.4, 0.3)$ are taken as the initial values of driving and reaction system respectively. The corresponding numerical result is depicted through Figure 4.

5 Function projective synchronization between different chaotic systems

Consider Lu system as the drive system as

$$D_t y_1 = a_2(y_2 - y_1) \tag{14}$$

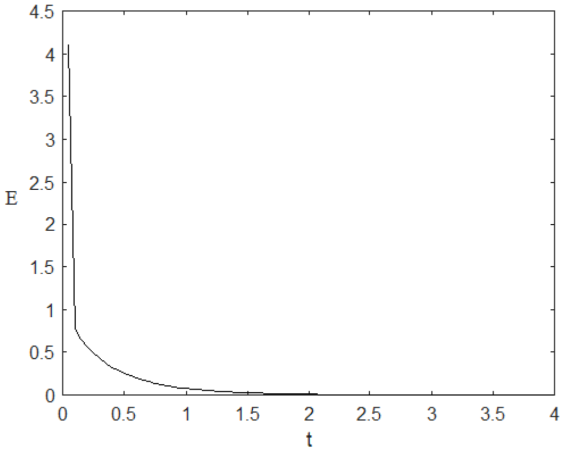

Fig. 4: The FPS outcome of equation (10) and equation (11)

$$D_t y_2 = c_2 y_2 - y_1 y_3 \tag{15}$$
$$D_t y_3 = y_1 y_2 - b_2 y_3 \tag{16}$$

The T-system is selected as response system as

$$D_t z_1 = a_3(z_2 - z_1) + v_1(t) \tag{17}$$
$$D_t z_2 = (c_3 - a_3)z_1 - a_3 z_1 z_3 + v_2(t) \tag{18}$$
$$D_t z_3 = -b_3 z_3 + z_1 z_2 + v_3(t) \tag{19}$$

Then, the error states become

$$e_1 = z_1 - k_1(T)y_1 \tag{20}$$
$$e_2 = z_2 - k_2(T)y_2 \tag{21}$$
$$e_3 = z_3 - k_3(T)y_3 \tag{22}$$

Using the appropriate controller, we obtain the fractional order error dynamics as

$$D_t E_z = (N_1(T,Z) + N_2(T,Z))E_z, \tag{23}$$

where

$$N_1(T,Z) = \begin{pmatrix} -2.6 & 2.6 & 0 \\ -25.4 + 2.6z_3 & -1 & -k_1(y_1) \\ k_1(y_1) & 0 & -0.4 \end{pmatrix}$$

and

$$N_2(T,Z) = \begin{pmatrix} 0 & -28 + 2.6z_3 & -z_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence, we obtain

$$N_1(T,Z) + N_2(T,Z) = \begin{pmatrix} -2.6 & 2.6 & 0 \\ -25.4 + 2.6z_3 & -1 & -k_1(y_1) \\ k_1(y_1) & 0 & -0.4 \end{pmatrix} + \begin{pmatrix} 0 & -28 + 2.6z_3 & -z_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

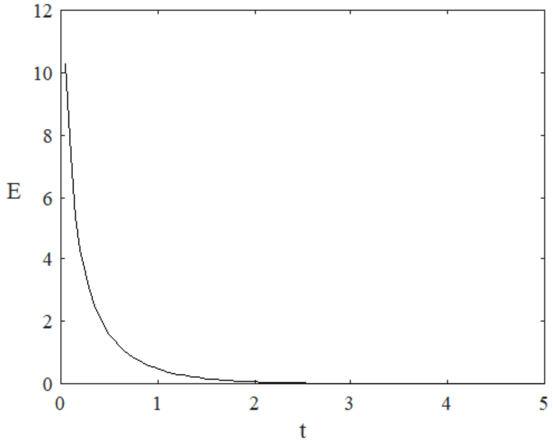

Fig. 5: The error dynamics of the FPS between the equation (15) and the equation (16)

$$= \begin{pmatrix} -2.6 & -25.4 + 2.6z_3 & -z_2 \\ -25.4 + 2.6z_3 & -1 & -k_1(y_1) \\ k_1(y_1) & 0 & -0.4 \end{pmatrix}$$

Choosing real symmetric positive definite matrix $P_2 = \text{diag}(1, 2, 1)$, we obtain

$$[M_1(T, Z) + N_2(T, Z)]P_2 + P_2[M_1(T, Z) + N_2(T, Z)]^T = \text{diag}(-5.2, -4, -0.8).$$

Choosing real symmetric positive definite matrix $Q_2 = \text{diag}(5.2, 4, 0.8)$, we get

$$[M_1(T, Z) + N_2(T, Z)]P_2 + P_2[M_1(T, Z) + N_2(T, Z)]^T = -Q_2.$$

Thus, the chaotic Lu system and T-system achieve the FPS.

5.1 Numerical simulation

Scaling function $\kappa(T) = \text{diag}(y_1 + y_3, y_2y_3, y_2 + y_3)$, and the error state $\mathbf{E}$ is

$$\mathbf{E} = \sqrt{\sum_{i=1}^{3} (z_i - k_i(T)y_i)^2}$$

The initial values of the equation (15) and the equation (16) are set at (3, 2, −1) and (4, −1, −6) respectively. The corresponding numerical result is depicted through Figure 5.

6 Conclusion

This article examined the chaotic systems of function projective synchronization (FPS) through tracking control method. Based on the stability criteria, this article has successfully represented FPS for identical Li system and also FPS between Lu system and T- system. Numerical simulations are performed to demonstrate the FPS design's efficacy and consistency in forecasting the correctness of theoretical results.

Declarations

- The authors received no specific funding for this study.
- The authors declare that they have no conflicts of interest to report regarding the present study.

- No Human subject or animals are involved in the research.
- All authors have mutually consented to participate.
- All the authors have consented the Journal to publish this paper.
- Authors declare that all the data being used in the design and production cum layout of the manuscript is declared in the manuscript.

References

[1] Pecora, L.M., Carroll, T.L.: Synchronization in chaotic systems. Physics Review Letters **64**, 821–824 (1990)

[2] Pecora, L.M., Carroll, T.L.: Driving systems with chaotic signals. Phys. Rev. A **44**, 2374–2383 (1991)

[3] Sun, J., Shen, Y., Yin, Q., Xu, C.: Compound synchronization of four memristor chaotic oscillator systems and secure communication. Chaos: An International Journal of Nonlinear Science **23**, 013140 (2013)

[4] Wu, A., Zhang, J.: Compound synchronization of fourth order memristor oscillator. Advances in Difference Equations, 100–106 (2014)

[5] Olusola, O.I., Vincent, U.E., Njah, A.N., Ali, E.: Control and synchronization of chaos in biological systems via backsteping design. International Journal of Nonlinear Science **11**, 121–128 (2011)

[6] Wang, L., Dong, T., Ge, M.-F.: Finite-time synchronization of memristor chaotic systems and its application in image encryption. Applied Mathematics and Computation **347**, 293–305 (2019)

[7] Liu, D., Zhu, S., Sun, K.: Global anti-synchronization of complex-valued memristive neural networks with time delays. IEEE Transactions on Cybernetics (2018). doi: 10.1109/TCYB.2018.2812708

[8] Wen, S., Zeng, Z., Huang, T., Chen, Y.: Fuzzy modeling and synchronization of different memristor-based chaotic circuits. Physics Letters A **377**, 2016–2021 (2013)

[9] Li, Z., Xu, D.: A secure communication scheme using projective chaos synchronization. Chaos, Solitons & Fractals **22**, 477–481 (2004)

[10] Chee, C.Y., Xu, D.: Secure digital communication using controlled projective synchronization of chaos. Chaos, Solitons & Fractals **23**, 1063–1070 (2005)

[11] Ghosh, D., Bhattacharya, S.: Projective synchronization of new hyperchaotic system with fully unknown parameters. Nonlinear Dynamics **61**, 11–21 (2010)

[12] Nian, F., Wang, X.: Projective synchronization of two different dimensional nonlinear systems. International Journal of Modern Physics B **27**, 1350113 (2013)

[13] Wu, X.J., Wang, H., Lu, H.T.: Modified generalized projective synchronization of a new fractional-order hyperchaotic system and its application to secure communication. Nonlinear Analysis: Real World Applications **13**, 1441–1450 (2012)

[14] Ouannas, A., Raghib, A.-S.: On matrix projective synchronization and inverse matrix projective synchronization for different and identical dimensional discrete-time chaotic systems.

Journal of Chaos, 1–7 (2016)

[15] Liu, F.: Matrix projective synchronization of chaotic systems and the application in secure communication. Applied Mechanics and Materials **644**, 4216–4220 (2014)

[16] Shi, Y., Wang, X., Zeng, X., Cao, Y.: Function matrix projective synchronization of non-dissipatively coupled heterogeneous systems with different-dimensional nodes. Advances in Difference Equations **198**, 1–12 (2019)

[17] Yadav, V.K., Shukla, V., Das, S., Leung, A.Y.T., Srivastava, M.: Function projective synchronization of fractional order satellite system and its stability analysis for incommensurate case. Chinese Journal of Physics **56**, 696–707 (2018)

[18] Yadav, V.K., Bhadauria, B.S., Das, S., Singh, A.K., Srivastava, M.: Stability analysis, chaos control of a fractional order chaotic chemical reactor system and its function projective synchronization with parametric uncertainties. Chinese Journal of Physics **55**, 594–605 (2016)

[19] Diethelm, K., Ford, J., Freed, A.: Detailed error analysis for a fractional adams method. Numerical Algorithms **36**, 31–52 (2004)

[20] Diethelm, K., Ford, J.: Multi-order fractional differential equations and their numerical solution. Applied Mathematics and Computation **154**, 621–640 (2004)

[21] Li, X.-F., Chlouverakis, K.E., Xu, D.-L.: Nonlinear dynamics and circuit realization of a new chaotic flow: A variant of lorenz, chen and lü. Nonlinear Analysis: Real World Applications **10**, 2357–2368 (2009)

[22] Lu, J., Chen, G., Zhang, S.: Dynamical analysis of a new chaotic attractor. International Journal of Bifurcation and Chaos in Applied Sciences and Engineering **12**, 1001–1015 (2002)

[23] Jiang, B., Hun, X., Bi, Q.: Hopf bifurcation analysis in the t system. Nonlinear Analysis: Real World Applications **11**, 522 (2010)

[24] Zhou, P., Zhu, W.: Function projective synchronization for fractional-order chaotic systems. Nonlinear Analysis: Real World Applications **12**, 811–816 (2011)