ON THE STRUCTURAL CONTROLLABILITY OF DYNAMIC SYSTEMS

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ABSTRACT

In dynamic systems, control engineers are interested in two important theoretical properties i.e. controllability and observability which are functions of all the entries of the matrices in the triple (A, B, C). In a practical application, we will actually have partial knowledge concerning the value of the entries in (A, B, C). Some values of (A,B,C) are known to be exactly zero, while other entries are known approximately (non zero entries). Under these conditions controllability cannot be determined by the usual method described in literature. In this paper structural controllability of a system is found without the complete knowledge of the entries of the matrices in the triple (A, B, C), which is a necessary condition for the controllability at any instance.

Key words- *structural, controllability, dynamic, systems*

1. Introduction

Consider the general abstraction of a dynamic system defined by its boundaries as shown in figure 1, where, $x(t) \in \mathbb{R}^n$ $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^r$ are the vectors of the system states, inputs and outputs respectively.

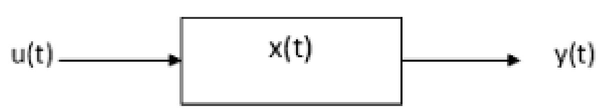


Figure 1. Input/Output representation of a dynamic system

When considering control of such a system, control engineers find a linear time-invariant model of the given dynamic system, which is represented in a general form as

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad (1)$$
$$y(t) = Cx(t)$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and. $C \in \mathbb{R}^{r \times n}$ of course, most real systems are nonlinear and have time-varying parameters [1]. However, a model described in (1), is often suitable for modeling small perturbations around a reference state and is therefore suitable for the design of a control system that attempts to reject disturbances and regulate the system close to the reference state [2].

2. Controllability and Observability

Control engineers are interested in the theoretical properties of system (1) , which in general are a function of the triple (A,B,C). The two important properties [3] are:

Controllability: can the inputs be chosen to drive (1) to steady state?

Observability: can the state vector be reconstructed from the measurements of the outputs?

The conditions for controllability and observability are based on the knowledge of numerical values of all the entries of the matrices in the triple (A, B, C). Evaluation of these conditions is thus subject to the following difficulties:

(a) The entries in (A,B,C) are typically only known approximately; uncertainties may arise from incomplete knowledge of the system, from approximation of nonlinear and time-varying properties not reflected by the linear model, assumptions do reduce the size of the model, or experimental error and noise in the estimation of the model parameters [4].

(b) Typically, the rank of the relevant matrices will have to be evaluated by numerical linear algebra on a computer. Due to rounding off the errors in computer arithmetic, the numerical estimation of the true rank of a matrix is also subject to a high degree of uncertainty.

Given the ambiguity entailed by use of the actual numerical entities, it is natural to investigate whether results concerning controllability and/or observability of a dynamic system can be derived from knowledge of the structure of the system alone [5]. In general, structural properties of a dynamic system are properties that are not dependent upon certain parameter values but hold for a large variety of parameter values.

3. Structure of a Dynamic System

In a practical application we will actually have partial knowledge concerning the precise value of the entries in (A, B, C). Some are known to be exactly zero (zero entries), while all the other entries are known approximately (non zero entries). Physically, this means we know a priori that certain inputs do not directly affect certain states, and similarly certain states do not affect certain outputs, etc. Hence the system has a certain structure that can be inferred solely from knowledge of how the variables influence each other.

Example: Consider the following linear time invariant model of a particular dynamic system (n = 3, m = 2, r = 2) [6]:
We can infer from the pattern of nonzero entries in

$$\dot{x} = \begin{bmatrix} -1 & 2 & 3 \\ 0 & -2 & 4 \\ 0 & 0 & -3 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u \qquad (2)$$

$$y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} x$$

these matrices, the following information about the physical system

(a) state 3 is only influenced by itself and the second input

(b) state 2 is influenced by state 3 and state 2 but no inputs etc….

This information will be known to the engineer while deriving a model for the system, even if he or she does not know the specific numerical values for the entries. (A, B, C) can be represented in incidence form as:

$$[A] = S_A = \begin{bmatrix} \times & \times & \times \\ 0 & \times & \times \\ 0 & 0 & \times \end{bmatrix}$$

$$[B] = S_B = \begin{bmatrix} \times & 0 \\ 0 & 0 \\ 0 & \times \end{bmatrix} \qquad (3)$$

$$[C] = S_C = \begin{bmatrix} \times & 0 & 0 \\ 0 & 0 & \times \end{bmatrix}$$

These matrices are also known as structure matrices of the dynamic system. The use of the notation [*A*] is to denote the structure matrix of *A*. The structure matrices (S_A S_B S_C) are not unique to the system (2). Any dynamic system with the same pattern of zero entries but different values for the non zero entries will have the same structure matrices [4]. It is therefore said that the structure matrices (S_A S_B S_C) define a class of structurally equivalent dynamic systems. A given structure matrix S_A defines the set of all admissible numerical matrices *A*(those with the same pattern of zero entries). Thus (S_A S_B S_C) describes a class (or a set) of structurally equivalent systems, given by

$$E(S_A, S_B, S_C) = \{(A,B,C) : [A] = S_A, [B] = S_B, [C] = S_C\} \qquad (4)$$

Hence, any real system in the set $E(S_A, S_B, S_C)$ is physically indistinguishable in a structural sense from the other members of the set. Clearly, given an incidence or structure matrix we can construct an equivalent graph. Use the following square structure matrix to construct the graph

$$[Q_0(E)] = \begin{bmatrix} [A] & [B] & 0 \\ 0 & 0 & 0 \\ [C] & 0 & 0 \end{bmatrix} \begin{matrix} x \\ u \\ y \end{matrix} \qquad (5)$$

where the vertices of the graph are partitioned into three sets- the input, the states, and the outputs. A directed graph $G([Q_0])$ is created according to the following rule: a directed edge from a vertex j to a vertex i if element q_{ij} of $[Q_0]$ is nonzero. Hence, the row of an entry gives the vertex which the edge is incident into, and the column gives the vertex, which the edge is incident out of (which is merely the transpose of the more conventional definition of adjacency matrix of a digraph). The graph $G([Q_0])$ for the class of structurally equivalent systems (2) is shown in figure 2. Note that a self loop occurs when a state variable influences itself. This is a very clear representation of the influence of various system variables on each other in the dynamic domain (or coupling). For example, it is clear that the inputs only indirectly affect x_2 through x_3.

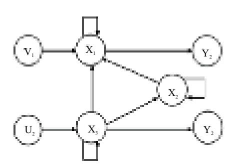


Figure 2 Digraph representation of the structure of a dynamic system

4. Structural Controllability

After representing the structure of large scale dynamic system consider the controllability and observability results based on purely structural considerations.

Definition structural controllability: A class of structurally equivalent systems E is said to be structurally controllable (s-controllable) if there exists at least one admissible realization (A, B, C) in E which is completely controllable.

Although these definitions are based on the existence of at least one admissible system that is controllable, it is a structural property. Further, almost all instances in E will turn out to be controllable. This approach allows us to derive results concerning a whole class of systems.

A result of these definitions is that structural controllability of the class E is a necessary condition for the complete controllability of any instance (A, B, C) $\in$ E. Therefore, all the members of a class E that is not structurally controllable will not be completely controllable, while on the other hand, structural controllability of the class E does not guarantee complete controllability of all its members.

In essence, from the point of view of system design, if we can determine that the structurally equivalent class to which our instance belongs, is not structurally controllable, then we can immediately exclude that instance from further consideration because we know that, no matter what we do, the system is not controllable. This prompts the use of these results for control system synthesis and design.

To employ this screening of alternatives, we need necessary and sufficient conditions for a class E of structurally equivalent systems to be s-controllable. From the point of view of controllability results alone consider the following square sub matrix (5) introduced earlier no: and its corresponding digraph.

$$[Q] = \begin{bmatrix} [A] & [B] \\ 0 & 0 \end{bmatrix}$$

Definition input connectability: A class of systems E is said to be input-connectable (or input-reachable) if in the corresponding graph *G([Q(E)])*, there exists for every state vertex a path from at least one of the input vertices.

Examine the graph in figure2, is it input connectable?

The necessary and sufficient conditions for a class of systems E to be s-controllable are:

(a) it is input-connectable and

(b) the structural rank of $[A:B] = n$

The necessity of these conditions can be demonstrated as shown-

(i) If a class of systems is not input-connectable, each instance in that class can be rearranged by row permutations to give the partition:

$$A = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}; \qquad B = \begin{pmatrix} 0 \\ B_2 \end{pmatrix}$$

where A_{11} is square. Note that the states in the top partition are not a function of the inputs, and are not coupled with the states in the lower partition. Clearly, the matrix $(B : AB \ldots\ldots A^{n-1} B)$ will be rank deficient for all members of the class.

(ii) If in contradiction with (b), the structural rank of $[A:B]$ is less than n, then for every instance in $(A,B) \in [A : B]$ $rank\ (A,B) < n$ and thus A has at least one eigenvalue equal to zero. Hence the criterion rank $(A-\lambda I : B) = n$ is violated for $\lambda = 0$ for all members of the class.

From a graph-theoretic point of view, it is easy to check for input-connectability (this is just a path tracing problem). The structural rank of [A: B] could be found by an algorithm that locates a maximum transversal. However, to state a purely graph-theoretic criterion for condition (ii) we must introduce the notion of static feedback control . In other words, we assume that we are able to measure the system states and feedback a linear time function of those states to define the control $u(t)=Ex(t)$ actions applied to the system.

Assume the matrix E is dense (i.e., we have a hypothetical multivariate controller in which all

$$[Q_1] = \begin{bmatrix} [A] & [B] \\ [E] & 0 \end{bmatrix}$$

measurements are used to manipulate all controls).

Consider with the following structure matrix:

for a system under static state feedback control. Introduce the graph-theoretic terms:

(a) a cycle family is a set of cycles that have no common vertex (vertex disjoint cycles)

(b) the width of a cycle family is then defined as the number of state vertices included in the cycle family.

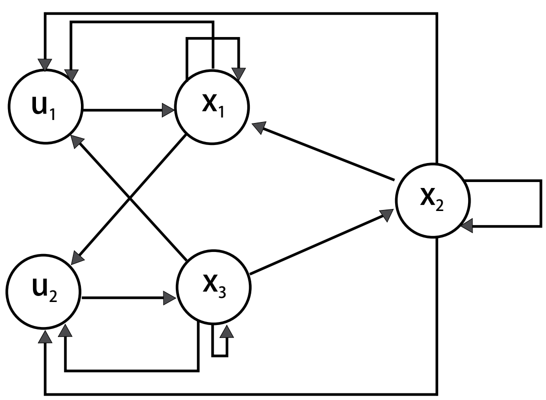
Introduce the following alternative to condition (i) above, which can be shown to be equivalent:

(ii) There is at least one cycle family of width n in *G([Q(E)])*

Now let us see if the example given in (2) is structurally controllable. The structural matrix of interest is:

$$[Q] = \begin{bmatrix} \times & \times & \times & \times & 0 \\ 0 & \times & \times & 0 & 0 \\ 0 & 0 & \times & 0 & \times \\ & \times & \times & 0 & 0 \\ \times & \times & \times & 0 & 0 \end{bmatrix}$$

and the equivalent graph is:



We have already established that the graph is input-connected, a candidate cycle family is $\{(u_2, x_3, x_2, u_2), (u_1, x_1, u_1)\}$ and the width of this family is 3. Alternatively, note that a transversal for [A: B] exists so the example system is at least structurally controllable.

5. Conclusion

Structural controllability of a dynamic system is very important and is a necessary condition for the complete controllability. It is relatively easy to introduce the dual notion of structural observability.

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