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On Radius of normalized Bessel functions

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Abstract

In this paper, we determine the radius of various starlikeness for the class of normalized Bessel functions of the first kind. We consider the certain subclassess of starlike functions such as the class associated with Booth lemniscate, the rational function $\boldsymbol{\psi(z) = 1+(z(k+z))/(k(k-z))}$, the class related to the sine function and modified sigmoid.

Keywords: Bessel function; starlike; convex; rational function; radius problems.

1 Introduction

The Bessel functions are an important class of mathematical functions that have a wide range of scientific and engineering applications. They play a significant role in the problems concerning wave propogation, diffraction and interference phenomena and heat condution in cylindrical coordinates. They are also used in modeling quantum mechanical systems particularly in solving the Schrödinger equation. The zeros of Bessel functions also play an important part in frequency modulation (FM) radio transmission, which is mathematically described by a harmonic distribution of a sine wave carrier modulated by a sine wave signal that may be represented using Bessel functions.

Let $\mathbb{D}(r) := \{z \in \mathbb{C} : |z| \leq r\}$ and $\mathbb{D} = \mathbb{D}(1)$. Let $\mathcal{A}$ be the class of analytic functions f in the open unit disk $\mathbb{D}$ and normalized by the condition $f(0) = 0$ and $f'(0) = 1$. Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ containing univalent functions. Let f and g be two analytic functions. Then f is said to be subordinate to g if there exist an analytic function ω in $\mathbb{D}$ with $\omega(0) = 0$ and $|\omega(z)| \leq 1$ for $z \in \mathbb{D}$ such that $f(z) = g(\omega(z))$. It is denoted by $f \prec g$. If g is univalent in $\mathbb{D}$, then f is subordinate to g if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$. Let ϕ be an analytic function in $\mathbb{D}$ with positive real part and whose range is starlike with respect to $\phi(0) = 1$ and $\phi'(0) > 0$. In 1992, Ma and Minda considered the class $S^*(\phi)$ of all $f \in \mathcal{A}$ satisfying $zf'(z)/f(z) \prec \phi(z)$. For different choices of ϕ the class $S^*(\phi)$ reduces to some well known subclasses of starlike function. For instance, when $\phi(z) = (1 + z)/(1 - z)$, $S^*(\phi)$ is the class S^* of starlike functions. Kargar *et al* [1] introduced the class $\mathcal{BS}^*(\alpha)$ associated with the Booth lemniscate. For $0 \leq \alpha < 1$, define $\mathcal{BS}^*(\alpha) = S^*(G_\alpha(z))$ where $G_\alpha(z) = 1 + z/(1 - \alpha z^2)$. The class $S^*_c := S^*((3 + 4z + 2z^2)/3)$ was studied in [2]. The class $S^*_R := S^*(\psi(z))$ is associated with the function $\psi(z) = 1 + (z(k + z))/(k(k - z))$ where $k = \sqrt{2} + 1$ was also made known in [3]. The class $S^*_q := S^*(z + \sqrt{1 + z^2})$ consists of lune starlike functions associated with the lune shaped region $\{w \in \mathbb{C} : 2|w| > |w^2 - 1|\}$ [4]. Cho *et al.* [5] introduced the class S^*_{sin} related to the function $1 + \sin z$. Recently, the class $S^*_{SG} := S^*(2/(1 + e^{-z}))$ associated with $\{w \in \mathbb{C} : |\log((2/(2 - w)) - 1)| < 1\}$ is also studied in [6]. For $\alpha \in \mathbb{R}$ the class $S^*_{\alpha,e}$ consists of all functions satisfying $zf'(z)/f(z) \prec \alpha + (1 - \alpha)e^z$. For $-1/(e - 1) \leq \alpha \leq e/(e - 1)$, the function $\alpha + (1 - \alpha)e^z$ is a function with positive real part and hence the class $S^*_{\alpha,e}$ consists of starlike univalent functions[7]. In this paper, we consider $0 \leq \alpha < 1$. The class $S^*_{3\mathcal{L}} := S^*(1 + 4z/5 + z^4/5)$ associated with convex combination of linear and exponential functions was studied in [8].

Let $\mathcal{M}$ be a set of functions in $\mathcal{A}$ and P be a property. The supremum R of all the radii so that each function $f \in \mathcal{M}$ has the property P in the disk $\mathbb{D}_r$, $0 \leq r \leq R$ is known as radius of property for the set $\mathcal{M}$, denoted by $R_P(\mathcal{M})$.

In 2014, Baricz and Szász [9, 10] obtained the radii of starlikeness and convexity for the Bessel functions of the first kind. Bohra *et al.* [11] determined the radius of starlikeness of order α and the k-uniformly convex radius of the normalized Bessel functions of the first kind. Recently, the authors [12] studied the radii of lemniscate starlikeness and Janowski starlikeness for certain normalizations of q-Bessel functions. For a more detailed study of Bessel function of the first kind one may refer to [13, 14]. Motivated by the aforementioned works, in this paper we determine the radius of various starlikeness for normalized Bessel functions of the first kind.

The Bessel function of the first kind of order ν is a particular solution of the second order homogeneous Bessel differential equation

$$z^2 w''(z) + zw'(z) + (z^2 - \nu^2)w(z) = 0, \ \nu \in \mathbb{C}.$$

This function has an infinite series expansion

$$J_\nu(z) = \sum_{n\geq 0} \frac{(-1)^n}{n!\Gamma(n + \nu + 1)} \left(\frac{z}{2}\right)^{2n+\nu},$$

where $z \in \mathbb{C}$ and $\nu \in \mathbb{C}$ such that $\nu \neq -1, -2, \cdots$. We consider the following three normalizations of the Bessel function of first kind:

$$f_\nu(z) = (2^\nu \Gamma(\nu + 1)J_\nu(z))^{1/\nu} = z - \frac{1}{4\nu(\nu + 1)} z^3 + \cdots, \ \nu \neq 0 \tag{1.1}$$

It is easily seen that the functions f_ν is normalized and analytic. Also $f_\nu(z) = \exp(\frac{1}{\nu}\log(2^\nu\Gamma(\nu+1)J_\nu(z)))$ where log represents the principal branch of the logarithm function. It is known that the zeros of the Bessel function J_ν are all real for $\nu > 0$. Thus the Bessel function admits Weierstrass decomposition [14, p. 498] of the form

$$J_\nu(z) = \frac{z^\nu}{2^\nu \Gamma(\nu + 1)} \prod_{n\geq 1} \left(1 - \frac{z^2}{j^2_{\nu,n}}\right) \tag{1.2}$$

where $j_{\nu,n}$ denotes the nth positive zero of the Bessel function J_ν. The zeros of a Bessel function satisfy the inequality $j_{\nu,1} < j_{\nu,2} < \cdots$ for $\nu > 0$ [15, p.235].

This infinite product is uniformly convergent on each compact subset of $\mathbb{C}$. From (1.2), we have

$$\frac{zJ'_\nu(z)}{J_\nu(z)} = \nu - \sum_{n\geq 1} \frac{2z^2}{j^2_{\nu,n} - z^2}. \tag{1.3}$$

Using (1.1) and (1.3), we get

$$\frac{zf'_\nu(z)}{f_\nu(z)} = \frac{1}{\nu}\frac{zJ'_\nu(z)}{J_\nu(z)} = 1 - \frac{1}{\nu}\sum_{n\geq 1} \frac{2z^2}{j^2_{\nu,n} - z^2}. \tag{1.4}$$

2 Main Result

In order to prove the main result we need the following lemmas.

Lemma 2.1. *[16, Lemma 3.2, p.310] For $|z| \le r < 1$, $|z_k| = R > r$ we have*

$$\left|\frac{z}{z - z_k} + \frac{r^2}{R^2 - r^2}\right| \le \frac{Rr}{R^2 - r^2}. \tag{2.1}$$

Lemma 2.2. *[17, Lemma 3.4, p.1392] Let $G_\alpha(z) = 1 + \dfrac{z}{1 - \alpha z^2}$ $(0 \le \alpha < 1)$. The inclusion relation is as follows:*

$$\left\{w \in \mathbb{C} : |w - 1| < \frac{1}{1 + \alpha}\right\} \subset G_\alpha(\mathbb{D}) \subset \left\{w \in \mathbb{C} : |w - 1| < \frac{1}{1 - \alpha}\right\}.$$

Lemma 2.3. *[2, Lemma 2.5, p.926] For $1/3 < a < 3$, let r_a be given by*

$$r_a = \begin{cases} (3a - 1)/3, & 1/3 < a \le 5/3; \\ 3 - a, & 5/3 \le a < 3 \end{cases} \tag{2.2}$$

and R_a be given by

$$R_a = \begin{cases} 3 - a, & 1/3 < a \le 11/9; \\ \sqrt{\dfrac{(3a - 1)^3}{27(a - 1)}}, & 11/9 \le a < 3 \end{cases} \tag{2.3}$$

Then

$$\{w \in \mathbb{C} : |w - a| < r_a\} \subseteq \Omega_C \subseteq \{w \in \mathbb{C} : |w - a| < R_a\}.$$

Lemma 2.4. *[3, Lemma 2.2 p. 202] For $2(\sqrt{2} - 1) < a < 2$, let r_a be given by*

$$r_a = \begin{cases} a - 2(\sqrt{2} - 1), & \text{if } 2(\sqrt{2} - 1) < a \le \sqrt{2} \\ 2 - a, & \text{if } \sqrt{2} \le a < 2 \end{cases} \tag{2.4}$$

then

$$\{w : |w - a| < r_a\} \subset \psi(\mathbb{D})$$

where $\psi(\mathbb{D}) = 1 + \dfrac{z}{k}\left(\dfrac{k + z}{k - z}\right) = 1 + \dfrac{1}{k}z + \dfrac{2}{k^2}z^2 + \cdots,\ k = \sqrt{2} + 1.$

Lemma 2.5. *[18, Lemma 2.1] For $\sqrt{2} - 1 < a \le \sqrt{2} + 1$, let $r_a = 1 - |\sqrt{2} - a|$ and $R_a = \sqrt{a^2 + 1}$. Then*

$$\{w : |w - a| < r_a\} \subset \left\{w : |w^2 - 1| < 2|w|\right\} \subset \{w : |w - a| < R_a\}.$$

Lemma 2.6. *[7, Lemma 2.2] For $\alpha + (1 - \alpha)e < a < \alpha + (1 - \alpha)e$, let r_a be given by*

$$r_a = \begin{cases} (a - \alpha) - (1 - \alpha)/e, & \alpha + (1 - \alpha)/e < a \le \alpha + (1 - \alpha)(e + e^{-1})/2; \\ e(1 - \alpha) - (a - \alpha), & \alpha + (1 - \alpha)(e + e^{-1})/2 \le a < \alpha + (1 - \alpha)e \end{cases} \tag{2.5}$$

and R_a be given by

$$R_a = \begin{cases} e(1 - \alpha) - (a - \alpha), & \alpha + (1 - \alpha)/e < a \le \alpha + (1 - \alpha)e/2; \\ \sqrt{z(\theta_\alpha)}, & \alpha + (1 - \alpha)e/2 \le a < \alpha + (1 - \alpha)e \end{cases} \tag{2.6}$$

Then,

$$\{w : |w - a| < r_a\} \subset \left\{w : \left|\log\left(\frac{w - \alpha}{1 - \alpha}\right)\right| < 1\right\} \subset \{w : |w - a| < R_a\}.$$

Lemma 2.7. *[5, Lemma 3.3] Let* $1-\sin 1 \leq a \leq 1+\sin 1$ *and* $r_a = \sin 1 - |a-1|$. *Then the following holds*

$$\{w : |w-a| < r_a\} \subseteq \Omega_S \subseteq \{w : |w-1| < \sinh 1\}.$$

Lemma 2.8. *[6, Lemma 2.2] Let* $2/(1+e) < a < 2e/(1+e)$. *If*

$$r_a = \frac{e-1}{e+1} - |a-1|$$

then $\{w : |w-a| < r_a\} \subset \Delta_{SG}$ *where* Δ_{SG} *is the image of* $\mathbb{D}$ *under the function* $G(z) = 2/(1+e^{-z})$.

Lemma 2.9. *[8, Lemma 2.2, p.175] For* $2/5 < a \leq 2$, *let* r_a *given by*

$$r_a = \begin{cases} a - 2/5, & \text{if } 2/5 < a \leq 1 \\ \sqrt{(a-7/5)^2 + a/5}, & \text{if } 1 \leq a < 51/35 \\ 2-a, & \text{if } 51/35 \leq a < 2 \end{cases} \tag{2.7}$$

If $\phi(z) = 1 + 4z/5 + z^4/5$, *then* $\{w : |w-a| < r_a\} \subset \phi(\mathbb{D})$.

Theorem 2.10. *Let* $\nu > 0$ *and let* $t_1 = j_{\nu,1}$ *denote the smallest positive zero of the Bessel function* J_ν. *For the function* f_ν *we have following radius estimates*

1. *Let* $\xi_1(r) = r(1+\alpha)J'_\nu(r) - \nu J_\nu(r)$. *Then the* $\mathcal{BS}^*(\alpha)$*-radius of the function* $f_\nu(z)$ *is the root* $r_1 \in (0, t_1)$ *of the equation* $\xi_1(r) = 0$ *where* $0 \leq \alpha < 1$.
2. *Let* $\xi_2(r) = 3rJ'_\nu(r) - \nu J_\nu(r)$. *Then the* $\mathcal{S}^*_C$*-radius of the function* $f_\nu(z)$ *is the root* $r_2 \in (0, t_1)$ *of the equation* $\xi_2(r) = 0$.
3. *Let* $\xi_3(r) = rJ'_\nu(r) - 2(\sqrt{2}-1)\nu J_\nu(r)$. *The* $\mathcal{S}^*_R$*-radius of the function* $f_\nu(z)$ *is the root* $r_3 \in (0, t_1)$ *of the equation* $\xi_3(r) = 0$.
4. *Let* $\xi_4(r) = rJ'_\nu(r) - (1-\sqrt{2})\nu J_\nu(r)$. *The* $\mathcal{S}^*_q$*-radius of the function* $f_\nu(z)$ *is the root* $r_4 \in (0, t_1)$ *of the equation* $\xi_4(r) = 0$.
5. *Let* $\xi_5(r) = erJ'_\nu(r) - (e\alpha + (1-\alpha))\nu J_\nu(r)$. *The* $\mathcal{S}^*_{\alpha,e}$*-radius of the function* $f_\nu(z)$ *is the root* $r_5 \in (0, t_1)$ *of the equation* $\xi_5(r) = 0$ *where* $0 \leq \alpha < 1$.
6. *Let* $\xi_6(r) = rJ'_\nu(r) + (\sin 1 - 1)\nu J_\nu(r)$. *The* $\mathcal{S}^*_{\sin}$*-radius of the function* $f_\nu(z)$ *is the root* $r_6 \in (0, t_1)$ *of the equation* $\xi_6(r) = 0$.
7. *Let* $\xi_7(r) = (e+1)rJ'_\nu(r) - 2\nu J_\nu(r)$. *The* $\mathcal{S}^*_{SG}$*-radius of the function* $f_\nu(z)$ *is the root* $r_7 \in (0, t_1)$ *of the equation* $\xi_7(r) = 0$.
8. *Let* $\xi_8(r) = 5rJ'_\nu(r) - 2\nu J_\nu(r)$. *The* $\mathcal{S}^*_{3\mathcal{L}}$*-radius of the function* $f_\nu(z)$ *is the root* $r_8 \in (0, t_1)$ *of the equation* $\xi_8(r) = 0$.

Proof. In view of Lemma 2.1, for $|z| = r < t_1$, we have

$$\left| \frac{zf'_\nu(z)}{f_\nu(z)} - \left(1 - \frac{2}{\nu}\sum_{n\geq 1} \frac{r^4}{j^4_{\nu,n} - r^4}\right) \right| \leq \frac{2}{\nu}\sum_{n\geq 1} \frac{j^2_{\nu,n} r^2}{j^4_{\nu,n} - r^4} \tag{2.8}$$

where $j_{\nu,n}$ denotes the nth positive zero of the Bessel function J_ν. It is evident that $a = 1 - \frac{2}{\nu}\sum_{n\geq 1} \frac{r^4}{j^4_{\nu,n} - r^4} < 1$. Using (1.4), we get

$$\left| \frac{zf'_\nu(z)}{f_\nu(z)} - 1 \right| = \left| \frac{1}{\nu}\sum_{n\geq 1} \frac{2z^2}{j^2_{\nu,n} - z^2} \right| \leq \frac{1}{\nu}\sum_{n\geq 1} \frac{2r^2}{j^2_{\nu,n} - r^2}.$$

1. It follows by Lemma 2.2 that the function f_ν belongs in the class $\mathcal{BS}^*(\alpha)$ if

$$\frac{2}{\nu}\sum_{n\geq 1}\frac{r^2}{j_{\nu,n}^2-r^2}\leq\frac{1}{1+\alpha}$$

using (1.3) we obtain $\xi_1(r)\leq 0$. Let $g(r)=\frac{1}{1+\alpha}-\frac{2}{\nu}\sum_{n\geq 1}\frac{r^2}{j_{\nu,n}^2-r^2}$. Then $\lim_{r\to 0} g(r)=1/(1+\alpha)>0$ and $\lim_{r\to t_1} g(r)=-\infty$. By the intermediate value theorem $g(r)$ has a root r_1 in $(0,t_1)$. Therefore, the desired $\mathcal{BS}^*(\alpha)$-radius r_1 lies in $(0,t_1)$.
2. Since $a<1$ thus we consider $1/3<a\leq 5/3$. In view of Lemma 2.3, the function f_ν belongs to the class $\mathcal{S}_C^*$ whenever

$$\frac{2}{\nu}\sum_{n\geq 1}\frac{j_{\nu,n}^2r^2}{j_{\nu,n}^4-r^4}\leq 1-\frac{2}{\nu}\sum_{n\geq 1}\frac{r^4}{j_{\nu,n}^4-r^4}-\frac{1}{3}$$

which gives $\xi_2(r)<0$. Let $g(r)=\frac{2}{3}-\frac{2}{\nu}\sum_{n\geq 1}\dfrac{r^2}{j_{\nu,n}^2-r^2}$. Then $\lim_{r\to 0} g(r)=2/3>0$ and $\lim_{r\to t_1} g(r)=-\infty$. It follows by the intermediate value theorem that $g(r)$ has a root $r_2\in(0,t_1)$. Therefore, the required $\mathcal{S}_C^*$- radius r_2 belongs to $(0,t_1)$.
3. It is enough to consider the case $2(\sqrt{2}-1)<a\leq\sqrt{2}$. It follows by Lemma 2.4 that the function f_ν belongs to the class $\mathcal{S}_R^*$ whenever

$$\frac{2}{\nu}\sum_{n\geq 1}\frac{j_{\nu,n}^2r^2}{j_{\nu,n}^4-r^4}\leq 1-\frac{2}{\nu}\sum_{n\geq 1}\frac{r^4}{j_{\nu,n}^4-r^4}-2(\sqrt{2}-1)$$

which gives $\xi_3(r)<0$. Let $g(r)=3-2\sqrt{2}-\frac{2}{\nu}\sum_{n\geq 1}\dfrac{r^2}{j_{\nu,n}^2-r^2}$. Then $\lim_{r\to 0} g(r)=0.1715>0$ and $\lim_{r\to t_1} g(r)=-\infty$. In view of the intermediate value theorem that $g(r)$ has a root $r_3\in(0,t_1)$. Therefore, the required $\mathcal{S}_R^*$- radius r_3 belongs to $(0,t_1)$.
4. Since $a<1$ thus in view of Lemma 2.5 that the function f_ν belongs to the class $\mathcal{S}_q^*$ whenever

$$\frac{2}{\nu}\sum_{n\geq 1}\frac{j_{\nu,n}^2r^2}{j_{\nu,n}^4-r^4}\leq 1-\frac{2}{\nu}\sum_{n\geq 1}\frac{r^4}{j_{\nu,n}^4-r^4}+1-\sqrt{2}$$

which implies $\xi_4(r)<0$. Let $g(r)=2-2\sqrt{2}-\frac{2}{\nu}\sum_{n\geq 1}\dfrac{r^2}{j_{\nu,n}^2-r^2}$. Then $\lim_{r\to 0} g(r)=0.585>0$ and $\lim_{r\to t_1} g(r)=-\infty$. It follows by the intermediate value theorem that $g(r)$ has a root $r_4\in(0,t_1)$. Therefore, the required $\mathcal{S}_q^*$- radius r_4 belongs to $(0,t_1)$.
5. It is enough to consider the case $\alpha+(1-\alpha)/e<a<\alpha+(1-\alpha)(e+e^{-1})/2$. In view of Lemma 2.6 that the function f_ν belongs to the class $\mathcal{S}_{\alpha,e}^*$ if

$$\frac{2}{\nu}\sum_{n\geq 1}\frac{j_{\nu,n}^2r^2}{j_{\nu,n}^4-r^4}\leq 1-\frac{2}{\nu}\sum_{n\geq 1}\frac{r^4}{j_{\nu,n}^4-r^4}-\alpha-(1-\alpha)/e$$

which implies $\xi_5(r) < 0$. Let $g(r) = 1 - \frac{2}{\nu}\sum_{n\geq 1}\dfrac{r^2}{j_{\nu,n}^2 - r^2} - \alpha - (1-\alpha)/e$. Then $\lim_{r\to 0} g(r) = (1-\alpha)(1-e^{-1}) > 0$ and $\lim_{r\to t_1} g(r) = -\infty$. It follows by the intermediate value theorem that $g(r)$ has a root $r_5 \in (0, t_1)$. Therefore, the required $\mathcal{S}^*_{\alpha,e}$- radius r_5 lies in $(0, t_1)$.

6. Since $a < 1$ thus in view of Lemma 2.7 that the function f_ν belongs to the class $\mathcal{S}^*_{\sin}$ whenever

$$\frac{2}{\nu}\sum_{n\geq 1}\frac{j_{\nu,n}^2 r^2}{j_{\nu,n}^4 - r^4} \leq 1 - \frac{2}{\nu}\sum_{n\geq 1}\frac{r^4}{j_{\nu,n}^4 - r^4} + \sin -1$$

which implies $\xi_6(r) < 0$. Let $g(r) = \sin -\frac{2}{\nu}\sum_{n\geq 1}\dfrac{r^2}{j_{\nu,n}^2 - r^2}$. Then $\lim_{r\to 0} g(r) = \sin 1 > 0$ and $\lim_{r\to t_1} g(r) = -\infty$. It follows by the intermediate value theorem that $g(r)$ has a root $r_6 \in (0, t_1)$. Therefore, the required $\mathcal{S}^*_{\sin}$- radius r_6 lies in $(0, t_1)$.

7. It is enough to consider the case $2/(1+e) < a < 1$. In view of Lemma 2.8 that the function f_ν belongs to the class $\mathcal{S}^*_{SG}$ whenever

$$\frac{2}{\nu}\sum_{n\geq 1}\frac{j_{\nu,n}^2 r^2}{j_{\nu,n}^4 - r^4} \leq 1 - \frac{2}{\nu}\sum_{n\geq 1}\frac{r^4}{j_{\nu,n}^4 - r^4} - \frac{2}{1+e}$$

which gives $\xi_7(r) < 0$. Let $g(r) = 1 - \frac{2}{\nu}\sum_{n\geq 1}\dfrac{r^2}{j_{\nu,n}^2 - r^2} - \frac{2}{1+e}$. Then $\lim_{r\to 0} g(r) = (e-1)/(e+1) > 0$ and $\lim_{r\to t_1} g(r) = -\infty$. By the intermediate value theorem $g(r)$ has a root $r_7 \in (0, t_1)$. Therefore, the required $\mathcal{S}^*_{SG}$- radius r_7 belongs to $(0, t_1)$.

8. Since $a < 1$ thus we consider $2/5 < a \leq 1$. In view of Lemma 2.9, the function f_ν belongs to the class $\mathcal{S}^*_{3L}$ whenever

$$\frac{2}{\nu}\sum_{n\geq 1}\frac{j_{\nu,n}^2 r^2}{j_{\nu,n}^4 - r^4} \leq 1 - \frac{2}{\nu}\sum_{n\geq 1}\frac{r^4}{j_{\nu,n}^4 - r^4} - \frac{2}{5}$$

which gives $\xi_8(r) < 0$. Let $g(r) = \frac{3}{5} - \frac{2}{\nu}\sum_{n\geq 1}\dfrac{r^2}{j_{\nu,n}^2 - r^2}$. Then $\lim_{r\to 0} g(r) = 3/5 > 0$ and $\lim_{r\to t_1} g(r) = -\infty$. It follows by the intermediate value theorem that $g(r)$ has a root $r_8 \in (0, t_1)$. Therefore, the required $\mathcal{S}^*_{3L}$- radius belongs to $(0, t_1)$.

□

3 Methodology

The idea is to determine the disk containing the image of the open unit disk under the mapping $zf'(z)/f(z)$

4 Conclusion

In addition to the results obtained in this paper other radius estimates can be determined for the class of starlike functions associated with the functions $1+(2/\pi^2)(\log((1-\sqrt{z})/(1+\sqrt{z})))^2$; $(1+z)^{1/2}$ and $1 + z - (z^3/3)$. Also, one can consider other normalizations of the Bessel functions of the first kind.

Declarations

- The authors received no specific funding for this study.

- The authors declare that they have no conflicts of interest to report regarding the present study.
- No Human subject or animals are involved in the research.
- All authors have mutually consented to participate.
- All the authors have consented the Journal to publish this paper.
- Authors declare that all the data being used in the design and production cum layout of the manuscript is declared in the manuscript.

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