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Some properties of semi-continuous function in topological space

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Abstract

The research paper aims to introduce the concept of a semi-continuous function in the topological spaces. Furthermore, we explore some fundamental properties of these semi-continuous functions as well as weak-functions, in relation to semi-open sets defined by N. Levine[2].Additionally, this work yields some new decomposition of generalized continuous functions.

Keywords: Open set, closed set, interior, closure, semi-open set, semi-closed set.

1 Introduction

In 1963, Levine [1, 2]introduced the concept of semi-open set in topological space. Following N. Levine's work on semi-open set, numerous mathematicians explored various aspects and introduced new type of open sets as well as generalized open sets. The fundamental open and closed sets hold significant importance in the field of topology. Furthermore, N. Biswas [3, 4]defined semi-closed sets in the year 1969 and established their basic properties. Subsequent, S. G. Crossley and S. K. Hildebrand[5–7]defined semi-closure of a set and P. Das[8] introduced the notion of semi-limit point of a subset within a topological space, further, investigating the semi-derived set of such a subset of a topological space[9, 10]. Consequently with the introduction of semi-open sets in 1963, the concept of continuity, also generalized to semi-continuity by the N. Levine [2], emerged as a crucial weaker form of continuity in topological spaces, playing a significant role within the field. Later, S. G. Crossley and S. K. Hildebrand [7]defined irresolute functions (also known as weak functions) by utilizing semi-open sets . These mathematicians in turn generalized the concept of continuous functions to weak functions in the topological spaces [11].

2 Preliminaries

For the entirety of this paper, $(X, \mathcal{J})$ always denotes topological spaces. It should be understood that no separation axioms are presumed, unless we specifically indicate otherwise[12]. When A is a subset of $(X, \mathcal{J})$, then $C_L(A)$ and $I_N(A)$ denote the closure and interior of the set A respectively. Topological spaces[13], open set[14], closed set[15], semi-open set[16] and neighborhood[17] are abbreviated as **t-spaces**, **o-set**, **c-set**, **semi o-set** and **nbd.**, respectively.

Definition 2.1 [2]: Let $(X_1, \mathcal{J}_1)$ and $(X_2, \mathcal{J}_2)$ be t-spaces. A function

$$f : (X_1, \mathcal{J}_1) \to (X_2, \mathcal{J}_2)$$

is said to be *semi-continuous* at a point $x \in X_1$ if for each o-set G containing $f(x)$ in X_2, there exists a semi o-set H containing x in X_1 such that

$$f(H) \subseteq G \quad \text{or} \quad H \subseteq f^{-1}(G).$$

Alternatively, the function

$$f : (X_1, \mathcal{J}_1) \to (X_2, \mathcal{J}_2)$$

is said to be *semi-continuous* if it is semi-continuous at each point of the set X_1.

2.1 Semi-continuous function in terms of o-sets

Definition 2.2 : [2] Let $(X_1, \mathcal{J}_1)$ and $(X_2, \mathcal{J}_2)$ be topological spaces. Then the function

$$f : (X_1, \mathcal{J}_1) \to (X_2, \mathcal{J}_2)$$

is said to be *semi-continuous* if for each o-set H in X_2, the inverse image $f^{-1}(H)$ is a semi o-set in X_1.

Alternatively, the function f is said to be semi-continuous if and only if the inverse image of any o-set is a semi o-set.

2.2 Semi-continuous function in terms of c-sets

Definition 2.3: Let $(X_1, \mathcal{J}_1)$ and $(X_2, \mathcal{J}_2)$ be topological spaces. Then the function

$$f : (X_1, \mathcal{J}_1) \to (X_2, \mathcal{J}_2)$$

is said to be *semi-continuous* if for each c-set F in X_2, the inverse image $f^{-1}(F)$ is a semi c-set in X_1.

In other words, the function f is said to be semi-continuous if and only if the inverse image of any c-set is a semi c-set.

Therefore, every continuous function

$$f : (X_1, \mathcal{J}_1) \to (X_2, \mathcal{J}_2)$$

is semi-continuous, but the converse is not necessarily true; i.e., a semi-continuous function f may fail to be continuous.

We demonstrate this with an example:

Let $X = Y = [0, 1]$ with the usual topology and consider a function $f : X \to Y$ defined by

$$f(x) = \begin{cases} 1, & 0 \leq x \leq \frac{1}{2}, \\ 0, & \frac{1}{2} < x \leq 1. \end{cases}$$

This function f is not continuous in $[0, 1]$, but f is semi-continuous in $[0, 1]$. Since the function f takes only two values, 0 and 1, the inverse image of an open subset of Y is either $\varnothing$, $[0, \frac{1}{2}]$, $(\frac{1}{2}, 1]$, or X.

Each of these sets is a semi-open set in X.

3 Main Result

Theorem 3.1. Let $(X_1, \mathcal{J}_1)$ and $(X_2, \mathcal{J}_2)$ be t-spaces. Then the necessary and sufficient condition for a function

$$f : (X_1, \mathcal{J}_1) \to (X_2, \mathcal{J}_2)$$

to be semi-continuous at a point $a \in X_1$ is that for each neighborhood H of $f(a)$ in X_2, $f^{-1}(H)$ is a semi-neighborhood of the point a.

Proof:

Necessary Part:
Let the function f be semi-continuous at a point $a \in X_1$, and let G be any neighborhood of $f(a)$. Then there exists a semi-neighborhood H of a such that

$$H \subseteq f^{-1}(G),$$

and $f^{-1}(G)$ contains a semi-neighborhood of the point a. Hence, $f^{-1}(G)$ is itself a neighborhood of the point a.

Sufficient Part:
Let it be that for each neighborhood G of $f(a)$, $f^{-1}(G)$ is a semi-neighborhood of the point a. Put $H = f^{-1}(G)$, then H is a semi-neighborhood of the point a and

$$f(H) = f(f^{-1}(G)) \subseteq G.$$

Hence, the function f is semi-continuous at the point $a \in X_1$.

Theorem 3.2: Let $(X_1, \mathcal{J}_1)$ and $(X_2, \mathcal{J}_2)$ be t-spaces. Then a necessary and sufficient condition for a function $f : (X_1, \mathcal{J}_1) \to (X_2, \mathcal{J}_2)$ to be semi-continuous at a point $a \in X_1$ is that for each open set G containing $f(a)$ in X_2, there exists a semi-open set H containing the point a such that $f(H) \subseteq G$.

Proof:

Necessary Part:
Let the function f be semi-continuous at a point $a \in X_1$ and let G be any open set containing $f(a)$ in X_2. Then G is a neighborhood of $f(a)$, so there exists a semi-neighborhood U containing the point a such that $f(U) \subseteq G$. Therefore, there exists a semi-open set H containing the point a such that $H \subseteq U$. Thus, $f(H) \subseteq f(U) \subseteq G$, so $f(H) \subseteq G$.

Sufficient Part:
Let the given condition be satisfied, and let G be any neighborhood of $f(a)$. Then there exists an open set O containing $f(a)$ such that $O \subseteq G$. By hypothesis, there exists a semi-open set H containing the point a such that $f(H) \subseteq O \subseteq G$. Hence, the function f is semi-continuous at the point $a \in X_1$.

Theorem 3.3: Let (X_1, J_1) and (X_2, J_2) be t-spaces. Then the necessary and sufficient condition for a function

$$f : (X_1, J_1) \to (X_2, J_2)$$

to be semi-continuous is that for each open set H in X_2, the inverse image $f^{-1}(H)$ is a semi-open set in X_1.

Or, the function f is semi-continuous if and only if the inverse image of every open set is a semi-open set.

Proof:

Necessary Part: Let the function f be semi-continuous, and let G be any open set in X_2.

- If $f^{-1}(G) = \varnothing$, then $f^{-1}(G)$ is trivially a semi-open set in X_1.
- If $f^{-1}(G) \neq \varnothing$, let $a \in f^{-1}(G)$ be arbitrary. Then $f(a) \in G$. Since G is an open set containing $f(a)$, it is a neighborhood of $f(a)$. Hence, $f^{-1}(G)$ is a semi-neighborhood of the point a.

Thus, $f^{-1}(G)$ is a semi-neighborhood of each of its points. Hence, $f^{-1}(G)$ is a semi-open set in X_1.

Sufficient Part: Suppose $f^{-1}(G)$ is a semi-open set in X_1 for every open set G in X_2. Let a be an arbitrary point of X_1. We need to show that f is semi-continuous at the point a.
Let V be any open set containing $f(a)$. Then $f^{-1}(V)$ is a semi-open set containing the point a. Let

$$U = f^{-1}(V)$$

Then U is a semi-open set containing a. Also,

$$f(U) = f(f^{-1}(V)) \subseteq V \Rightarrow f(U) \subseteq V.$$

Hence, the function f is semi-continuous at the point a.
Since the function f is semi-continuous at an arbitrary point a of the set X_1, it follows that f is semi-continuous over X_1.

Theorem 3.4. Let $(X_1, \mathcal{J}_1)$ and $(X_2, \mathcal{J}_2)$ be t-spaces. Then a necessary and sufficient condition for a function

$$f : (X_1, \mathcal{J}_1) \to (X_2, \mathcal{J}_2)$$

to be semi-continuous is that for each c-set F in X_2, the inverse image $f^{-1}(F)$ is a semi c-set in X_1.

In other words, the function f is semi-continuous if and only if the inverse image of any c-set is a semi c-set.

Proof:

Since a set F is a c-set if and only if F^c is an o-set.

Necessary part: Suppose the function f is semi-continuous, and let F be any c-set in X_2. Then F^c is an o-set in X_2, so $f^{-1}(F^c)$ is a semi o-set in X_1. But

$$f^{-1}(F^c) = \left(f^{-1}(F)\right)^c.$$

Hence, $\left(f^{-1}(F)\right)^c$ is a semi o-set in X_1. Thus, $f^{-1}(F)$ is a semi c-set in X_1.

Sufficient part: Suppose that for every c-set F in X_2, $f^{-1}(F)$ is a semi c-set in X_1, and let G be any o-set in X_2. Then $f^{-1}(G^c) = \left(f^{-1}(G)\right)^c$ is a semi c-set in X_1. So, $f^{-1}(G)$ is a semi o-set in X_1. Hence, the function f is semi-continuous.

Theorem 3.5.
Let $\mathcal{J}_1$ and $\mathcal{J}_2$ be two topologies on a set X. Then the identity function $I : (X, \mathcal{J}_1) \to (X, \mathcal{J}_2)$ defined by $I(x) = x,\ \forall x \in X$ is semi-continuous if and only if $\mathcal{J}_1$ is finer than $\mathcal{J}_2$, i.e., $\mathcal{J}_2 \subseteq \mathcal{J}_1$.

Proof.
The function I is semi-continuous if and only if for every open set $G \in \mathcal{J}_2$, the inverse image $I^{-1}(G)$ is a semi-open set in $\mathcal{J}_1$.

Since $I^{-1}(G) = G$ and G is an open set, and every open set is also a semi-open set, it follows that $I^{-1}(G)$ is a semi-open set in $\mathcal{J}_1$.

Hence, the function I is semi-continuous if and only if $G \in \mathcal{J}_2 \Rightarrow G \in \mathcal{J}_1$, i.e., if and only if $\mathcal{J}_2 \subseteq \mathcal{J}_1$. □

Theorem 3.6. Let $(A, \mathcal{J}_A)$ be a subspace of the t-space $(X, \mathcal{J})$. Then the inclusion map

$$f = I : (A, \mathcal{J}_A) \to (X, \mathcal{J})$$

defined by $I(x) = x$, $\forall x \in A$, is semi-continuous.

Proof. Let G be any open subset of X. Then

$$I^{-1}(G) = A \cap G$$

is a semi-open set in A, since A is a subspace of X. Hence, the inclusion map

$$f = I : (A, \mathcal{J}_A) \to (X, \mathcal{J})$$

defined by $I(x) = x$, $\forall x \in A$, is semi-continuous.

Theorem 3.7.
Let $f : (X, \mathcal{J}_1) \to (Y, \mathcal{J}_2)$ be semi-continuous and let $A \subseteq X$. Then the restriction of f to A, written as f_A, defined by

$$f_A(x) = f(x) \quad \forall x \in A$$

is also semi-continuous concerning the relative topology on the set A.

Proof.
Let G be any open subset of Y. Then

$$f_A^{-1}(G) = A \cap f^{-1}(G)$$

Since $f^{-1}(G)$ is a semi-open set in X and A is a subspace of X, it follows that $f_A^{-1}(G)$ is a semi-open set in A with respect to the relative topology.

Therefore, the function f_A defined by $f_A(x) = f(x)$ for all $x \in A$ is also semi-continuous with respect to the relative topology on A. □

Theorem 3.8. Let $(X_1, \mathcal{J}_1)$ and $(X_2, \mathcal{J}_2)$ be t-spaces. Let $f : (X_1, \mathcal{J}_1) \to (X_2, \mathcal{J}_2)$ be a semi-continuous function. Then the semi-continuity of f is not destroyed if we replace the topology $\mathcal{J}_1$ of X_1 by a finer topology $\mathcal{J}_1^*$ or the topology $\mathcal{J}_2$ of X_2 by a weaker topology $\mathcal{J}_2^*$.

Proof. Let $f : (X_1, \mathcal{J}_1) \to (X_2, \mathcal{J}_2)$ be semi-continuous. So for every open set $G \in \mathcal{J}_2$, the pre-image $f^{-1}(G)$ is a semi-open set in $(X_1, \mathcal{J}_1)$.

Now, if $\mathcal{J}_1 \subseteq \mathcal{J}_1^*$, then $f^{-1}(G)$ remains a semi-open set in $(X_1, \mathcal{J}_1^*)$. Hence, the function $f : (X_1, \mathcal{J}_1^*) \to (X_2, \mathcal{J}_2)$ is semi-continuous. Thus, the semi-continuity of f is not destroyed if we replace the topology $\mathcal{J}_1$ by a finer topology $\mathcal{J}_1^*$.

Again, consider every open set $G \in \mathcal{J}_2^*$ and suppose that $\mathcal{J}_2^* \subseteq \mathcal{J}_2$, i.e., $\mathcal{J}_2^*$ is weaker than $\mathcal{J}_2$. Then $G \in \mathcal{J}_2$. Since $f : (X_1, \mathcal{J}_1) \to (X_2, \mathcal{J}_2)$ is semi-continuous and $G \in \mathcal{J}_2$, we have that $f^{-1}(G)$ is semi-open in $(X_1, \mathcal{J}_1)$.

Hence, $f : (X_1, \mathcal{J}_1^*) \to (X_2, \mathcal{J}_2^*)$ is semi-continuous because:

$$G \in \mathcal{J}_2 \Rightarrow G \in \mathcal{J}_2^* \Rightarrow f^{-1}(G) \text{ is semi-open in } \mathcal{J}_1 \Rightarrow f^{-1}(G) \text{ is semi-open in } \mathcal{J}_1^*.$$

□

Theorem 3.9: Let (X_1, J_1) and (X_2, J_2) be t-spaces, and let $f : (X_1, J_1) \to (X_2, J_2)$ be a function. Let $A \subseteq X_1$ and $B \subseteq X_2$. Then the following statements are equivalent:

1. f is semi-continuous.
2. The inverse image of each c-set in X_2 is a semi c-set in X_1.
3. $f[\text{int}(\text{cl}(A))] \subseteq \text{cl}(f(A))$.
4. $\text{int}(\text{cl}(f^{-1}(B))) \subseteq f^{-1}(\text{cl}(B))$.

Proof:

(1) ⇔ (2):

Assume f is semi-continuous and let F be any c-set in X_2. Then $X_2 - F$ is an o-set, so $f^{-1}(X_2 - F)$ is a semi o-set in X_1 by the definition of semi-continuity. But

$$f^{-1}(X_2 - F) = X_1 - f^{-1}(F),$$

so $f^{-1}(F)$ is a semi c-set in X_1. This proves that (1) ⇒ (2).

Conversely, suppose the inverse image of every c-set in X_2 is a semi c-set in X_1. Let G be any o-set in X_2, then $X_2 - G$ is a c-set in X_2, so

$$f^{-1}(X_2 - G) = X_1 - f^{-1}(G)$$

is a semi c-set in X_1, hence $f^{-1}(G)$ is a semi o-set. Thus, f is semi-continuous. Therefore, (2) ⇒ (1). Hence, (1) ⇔ (2).

(1) ⇔ (3):

Assume f is semi-continuous. Let $A \subseteq X_1$ and $\text{cl}(f(A))$ be a c-set in X_2. Then $f^{-1}(\text{cl}(f(A)))$ is a semi c-set in X_1, and since $f(A) \subseteq \text{cl}(f(A))$, we have

$$A \subseteq f^{-1}(\text{cl}(f(A))). \tag{1}$$

Since $f^{-1}(\text{cl}(f(A)))$ is semi c-set, it follows that

$$\text{int}(\text{cl}(A)) \subseteq \text{int}(\text{cl}(f^{-1}(\text{cl}(f(A))))) \subseteq f^{-1}(\text{cl}(f(A))). \tag{2}$$

Applying f gives

$$f[\text{int}(\text{cl}(A))] \subseteq \text{cl}(f(A)),$$

which proves (1) ⇒ (3).

Conversely, suppose $f[\text{int}(\text{cl}(A))] \subseteq \text{cl}(f(A))$ for all $A \subseteq X_1$. Let F be any c-set in X_2 and set $A = f^{-1}(F)$ so $f(A) \subseteq F$. Since F is closed, $\text{cl}(F) = F$, so

$$f[\text{int}(\text{cl}(A))] \subseteq \text{cl}(f(A)) \subseteq \text{cl}(F) = F.$$

Thus,

$$\text{int}(\text{cl}(A)) \subseteq f^{-1}(F) = A,$$

so A is a semi c-set. Therefore, $f^{-1}(F)$ is semi c-set for all c-sets F in X_2, hence f is semi-continuous. Thus, (3) $\Rightarrow$ (1). Therefore, (1) $\Leftrightarrow$ (3).

(1) $\Leftrightarrow$ (4):

Assume f is semi-continuous. Let $B \subseteq X_2$, then $\text{cl}(B)$ is a c-set in X_2, so $f^{-1}(\text{cl}(B))$ is a semi c-set in X_1, and since $B \subseteq \text{cl}(B)$, we have

$$f^{-1}(B) \subseteq f^{-1}(\text{cl}(B)). \tag{3}$$

Taking closure and interior gives

$$\text{int}(\text{cl}(f^{-1}(B))) \subseteq \text{int}(\text{cl}(f^{-1}(\text{cl}(B)))) \subseteq f^{-1}(\text{cl}(B)).$$

This proves (1) $\Rightarrow$ (4).

Conversely, suppose

$$\text{int}(\text{cl}(f^{-1}(B))) \subseteq f^{-1}(\text{cl}(B))$$

for every $B \subseteq X_2$. Let F be a c-set in X_2, so $\text{cl}(F) = F$. Set $B = F$, then

$$\text{int}(\text{cl}(f^{-1}(F))) \subseteq f^{-1}(F),$$

so $f^{-1}(F)$ is a semi c-set. Hence, f is semi-continuous. This proves (4) $\Rightarrow$ (1). Therefore, (1) $\Leftrightarrow$ (4).

■

Declarations

- The authors received no specific funding for this study.
- The authors declare that they have no conflicts of interest to report regarding the present study.
- No Human subject or animals are involved in the research.
- All authors have mutually consented to participate.
- All the authors have consented the Journal to publish this paper.
- Authors declare that all the data being used in the design and production cum layout of the manuscript is declared in the manuscript.

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